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Crossover from Orthogonal to Unitary Symmetry for Ballistic Electron Transport in Chaotic Microstructures

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We study the ensemble-averaged conductance as a function of applied magnetic field for ballistic electron transport across few-channel microstructures constructed in the shape of classically chaotic billiards. We analyse the results of recent experiments, which show suppression of weak localization due to magnetic field, in the framework of random-matrix theory. By analysing a random-matrix Hamiltonian for the billiard–lead system with the aid of Landauer's formula and Efetov's supersymmetry technique, we derive a universal expression for the weak-localization contribution to the mean conductance that depends only on the number of channels and the magnetic flux. We consequently gain a theoretical understanding of the continuous crossover from orthogonal symmetry to unitary arising from the violation of time-reversal invariance for generic chaotic systems. © 1995 Academic Press, Inc.

1. INTRODUCTION

Random-matrix theory has proven to be a very successful tool for the understanding of localization phenomena in disordered diffusive systems, for example,

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universal conductance flutuations [1, 2], Altshuler–Aronov–Spivak oscillations [3, 4], and persistent currents [5]. In a number of recent experiments, weak-localization effects have been observed in ballistic cavities [6–8] in which electron scattering from the boundaries is classically chaotic. It is interesting to also investigate the application of random-matrix theory to such problems as one can expect it to be able to address questions not accessible to other methods, such as semiclassical analysis.

In this work, we look at ballistic electron transport through a microstructure that has the shape of a classically chaotic billiard, in the presence of an applied magnetic field B. The typical setup for a stadium is shown in Fig. 1. The stadium, whose length L is less than the mean free path l and the phase-coherence length L_φ, has attached to it two asymmetrically positioned ideal quasi-one-dimensional leads which carry the incoming and outgoing electron transverse modes. We assume an equal number, $M/2$, of incoming channels (labelled by index a) and outgoing channels (labelled by index b). Thus M counts the total number of open channels coupled to the stadium.

We use this physical system to study the breaking of time-reversal symmetry, as measured by the conductance $G(B) = (e^2/h)\, g(B)$, as the magnetic field B is turned on. The relevant experimental observation [6–8] is that the average dimensionless conductance $\bar{g}$ dips at zero magnetic field $B = 0$. The $\bar{g}$ versus B curve is illustrated

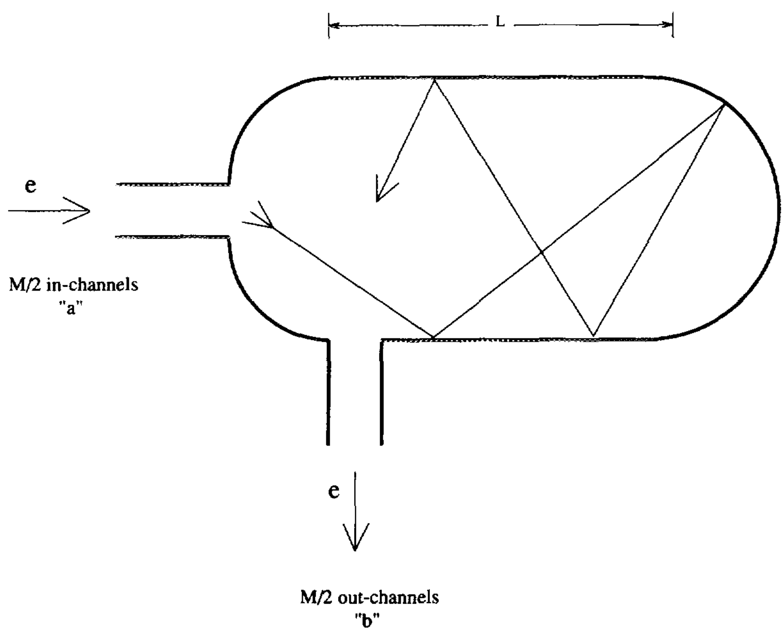

FIG. 1. Representation of a typical lead-stadium configuration. The size of the stadium L is small compared with the elastic mean free path l, ensuring that the electron motion is ballistic. A total of M external channels couple to the stadium through the two leads.

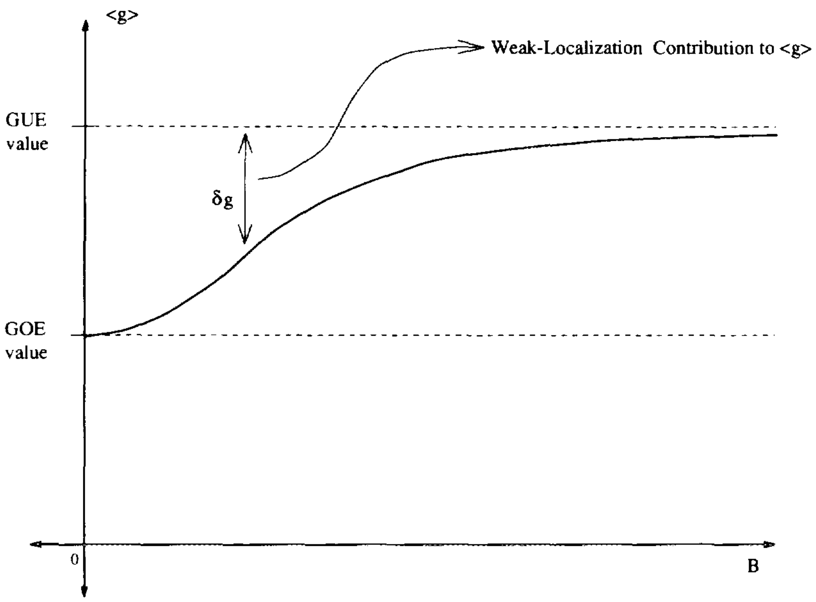

FIG. 2. Schematic depiction of the zero-field dip in the average magnetoconductance associated with weak localization. The curve interpolates between values associated with pure GOE and GUE for low and high magnetic fields B, respectively.

schematically in Fig. 2. The conductance interpolates between a minimum value, which can be associated with the Gaussian orthogonal ensemble (GOE), at $B=0$ and an asymptotic upper value, which can be associated with the Gaussian unitary ensemble (GUE), for large B. The difference δg between the upper GUE limit and the mean dimensionless conductance $\bar{g}$ represents the weak-localization contribution to $\bar{g}$. The suppression of weak localization due to non-zero magnetic field, i.e., the fact that $\delta g \to 0$ as $B \to \infty$, corresponds to a GOE→GUE transition.

The appearance of the orthogonal and unitary random-matrix ensembles in this discussion arises from the fact that the fluctuation properties of the quantum spectra for closed chaotic billiards are the same as those of (i) the GOE for $B=0$ and (ii) the GUE for B sufficiently different from zero. The former, being an ensemble of Gaussion distributed real-symmetric matrices, respects time-reversal invariance, while the latter, being an ensemble of Gaussian distributed Hermitian matrices, does not.

The presence and magnitude of the weak-localization term in $\bar{g}(B)$ at $B=0$ can easily be understood in the framework of random-matrix theory (without detailed calculation) as a consequence of (i) elastic enhancement (or equivalently, coherent back-scattering), derived in Section 5.1, and (ii) unitarity of the S-matrix. Our starting point is the connection between the conductance g and the S-matrix given by Landauer's formula:

$$g = \sum_{a=1}^{M/2} \sum_{b=M/2+1}^{M} \{ |S_{ab}|^2 + |S_{ba}|^2 \}, \tag{1.1}$$

where a factor of 2 for spin is effectively included in this expression. Here, the index a sums over the $M/2$ incoming channels and b sums over the $M/2$ outgoing channels, so that the summation in Eq. (1.1) runs over two non-overlapping sets of channels $\{a\} \neq \{b\}$. If we assume all channels to be equivalent, so that all mean values $\overline{|S_{ab}|^2}$ are equal for $a \neq b$, then we obtain

$$\bar{g} = 2\left(\frac{M}{2}\right)\left(\frac{M}{2}\right)\overline{|S_{ab}|^2} = \frac{M^2}{2}\overline{|S_{ab}|^2}. \tag{1.2}$$

Now, according to (i), we have

$$\overline{|S_{cc}|^2} = \kappa\,\overline{|S_{cc'}|^2} \qquad \text{for} \quad c \neq c' \tag{1.3}$$

with

$$\kappa = \begin{cases} 2 & \text{for GOE} \\ 1 & \text{for GUE.} \end{cases} \tag{1.4}$$

As a consequence of (ii), we have

$$\sum_{c,c'=1}^{M} \overline{|S_{cc'}|^2} = \mathrm{tr}\,\mathbf{1}_M = M. \tag{1.5}$$

Combination of the two equations (1.3) and (1.5) yields

$$\begin{aligned} M &= \sum_{c=1}^{M} \overline{|S_{cc}|^2} + \sum_{\substack{c,c'=1 \\ c \neq c'}}^{M} \overline{|S_{cc'}|^2} \\ &= M\kappa\,\overline{|S_{ab}|^2} + M(M-1)\overline{|S_{ab}|^2} \end{aligned} \tag{1.6}$$

for any $a \neq b$. Therefore, we see that

$$\overline{|S_{ab}|^2} = \frac{1}{M-1+\kappa} = \begin{cases} 1/M & \text{for GUE} \quad (\kappa = 1) \\ 1/(M+1) & \text{for GOE} \quad (\kappa = 2). \end{cases} \tag{1.7}$$

We can now use Eq. (1.7) to look at the expansions of the corresponding values of $\bar{g}$ in powers of M^{-1}. For the pure GUE (large magnetic field), we obtain

$$\bar{g}_{\mathrm{GUE}} = M/2, \tag{1.8}$$

while for the pure GOE ($B=0$), we see that

$$\bar{g}_{\mathrm{GOE}}=\frac{M^2}{2(M+1)}$$
$$=\bar{g}_{\mathrm{GUE}}-\frac{1}{2}+O\left(\frac{1}{M}\right). \tag{1.9}$$

The contribution $-\frac{1}{2}$ to the RHS of the equation above represents the weak-localization term in the conductance and gives the magnitude of the gap between the upper and lower asymptotic values in Fig. 2 in the limit of large channel number M. Weak localization is absent for the GUE, evidently being suppressed by the magnetic field.

One should also observe that, for general values of M, the weak-localization term has the value $M/(2(M+1))$. It increases from $\frac{1}{3}$ for $M=2$ to $\frac{1}{2}$ for $M\to\infty$. Nonetheless, the weak-localization correction decreases relative to $\bar{g}$ which is linear in M. All this agrees with recent results obtained [9] for the circular unitary (CUE) and circular orthogonal (COE) ensembles. The agreement is not surprising: It has been shown [10] that the GOE and the COE distributions for the S-matrix elements coincide when the coupling with the leads is maximized. Similar arguments apply to the unitary case.

We now consider the question: How does a finite magnitude field interpolate between $\bar{g}_{\mathrm{GOE}}$ ($B=0$) and $\bar{g}_{\mathrm{GUE}}$ (B sufficiently large)? Semiclassical analysis [12] implies that $\delta g(B)=\bar{g}_{\mathrm{GUE}}-\bar{g}(B)$ is a Lorentzian. However, this work was only able to include the diagonal contributions (i.e., taking into account only interference between symmetry-related classical paths) and was not able to achieve quantitative agreement with fully quantal numerical simulations. In view of this, one should ask whether the full result remains Lorentzian.

Our aim in this investigation is to understand the generic features of the continuous GOE → GUE crossover as a function of magnetic field B for *any* classically chaotic microstructure in the framework of random-matrix theory. From the semiclassical perspective, our random-matrix approach may be viewed as taking into account the generic properties described by the long classical trajectories, in which system-specific information is washed out. Therefore, we shall need to restrict attention to billiard/lead configurations that are dominated by long classical trajectories. Our results will not take account of effects associated with the specific geometry of a billiard, which would be probed by short trajectories. Moreover, the density of long trajectories is large—growing exponentially with the period—and therefore virtually impossible to account for in the framework of semiclassical theory.

On an energy (rather than a time) scale, such generic features are displayed on intervals measured in units of the mean level spacing d. The suppression of weak localization becomes effective whenever the matrix elements of the interaction of an electron with the magic field become comparable with d. This is why we believe that our approach should yield results which are relevant for the problem. There exists a time scale—the mixing time τ_{mix} defined below—which relates to the strength of these matrix elements. The suppression of weak localization will be seen to depend actually on the competition between this time scale and a second time scale τ_{dec}, the mean life-time of the levels in the billiard. The associated width $\Gamma_{\text{dec}} = h/\tau_{\text{dec}}$ grows linearly with the number of channels. Our final result essentially depends on the ratio of these two time scales.

The paper is set out as follows: In Section 2, we define the model and discuss the values of its parameters. Section 3 deals with relating the parameter that controls the GOE → GUE crossover to the strength of the external magnetic field. In Section 4, we outline the supersymmetry formalism used to extract an exact result for the ensemble-averaged conductance and explicitly construct the generating fucntion that is central to this approach. We also summarize the steps leading to the final analytical expression for $\bar{g}$, which is then displayed and discussed in Section 5. Results of the numberical evaluation of the magnetic field and channel number dependence of the weak-localization term are presented in Section 6, along with a discussion of the Lorentzian approximation. Section 7 contains some interpretation of our results, as well as a comparison with experimental data. Finally, we present out conclusions in Section 8. A few appendices contain various technical details.

2. The Model

We first consider an electron moving in a closed billiard, whose Hamiltonian H we model in terms of an ensemble of random matrices. We then take account of the presence of external channels by introducing a coupling between H and the leads. This allows us to define a scattering matrix which we relate to the conductance via Landauer's formula, viz. Eq. (1.1). We shall maximize the coupling between H and the leads. Then, the (ensemble-averaged) conductance $\bar{g}(t, M)$ depends only on the strength of the magnetic field B (expressed in terms of a dimensionless parameter t), and on the total number M of channels. Using ergodicity, we compare this quantity with experiment. We neglect the Coulomb interaction between electrons as well as all other dephasing effects.

For the Hamiltonian of the closed billiard, we consider an ensemble of random matrices $H_{\mu\nu}$ of dimension N, given by [11]

$$H_{\mu\nu} = H^{(S)}_{\mu\nu} + i\sqrt{(t/N)}\, H^{(A)}_{\mu\nu}. \tag{2.1}$$

The ensemble $H^{(S)}$ represents a GOE with zero mean and whose non-diagonal elements have variance λ^2/N, while $H^{(A)}$ denotes an ensemble of Gaussian distributed real antisymmetric matrices whose non-vanishing elements have zero mean, the same variance λ^2/N, and are uncorrelated for $\mu > \nu$. Explicitly,

$$\begin{aligned} \overline{H^{(S)}_{\mu\nu} H^{(S)}_{\mu'\nu'}} &= \frac{\lambda^2}{N} (\delta_{\mu\mu'}\delta_{\nu\nu'} + \delta_{\mu\nu'}\delta_{\nu\mu'}), \\ \overline{H^{(A)}_{\mu\nu} H^{(A)}_{\mu'\nu'}} &= \frac{\lambda^2}{N} (\delta_{\mu\mu'}\delta_{\nu\nu'} - \delta_{\mu\nu'}\delta_{\nu\mu'}). \end{aligned} \tag{2.2}$$

The parameter λ fixes the mean level spacing d of the GOE and must, in principle, be adjusted to the local mean level spacing at the Fermi energy of the billiard at hand. In the end, we shall take the limit $N \to \infty$, in which case the average level density of the ensemble (2.1) will take the form of Wigner's semicircle. We shall also normalize all energies such that the Fermi energy is zero, $E_F = 0$, in which case it sits at the centre of the Wigner semicircle. The mean level spacing at the Fermi energy is then given by $d = \pi\lambda/N$.

To estimate the scale of the GOE $\to$ GUE crossover, we note that this transition will take place when the typical matrix element of the time-reversal violating perturbation becomes comparable with the mean GOE level spacing,

$$\sqrt{t/N} \cdot \sqrt{\lambda^2/N} \sim \lambda/N, \tag{2.3}$$

which implies $t \sim 1$, i.e., for a value of t of order unity on the scale of N. This is why we chose to include a factor $N^{-1/2}$ in front of $H^{(A)}$ in Eq. (2.1).

The coefficient of $iH^{(A)}$ gives the strength of the time-reversal violating perturbation and is expected to be proportional to the magnetic field, since the variation of the levels of the stadium with magnetic field is linear in the flux $\phi = BA$ through the area A of the stadium. It is convenient to express $\sqrt{t} = k\phi/\phi_0$, where $\phi_0 = h/e$ is the fundamental flux quantum, and k is then a number of order unity. An estimate of k can be provided by computing the Δ_3-statistic of a closed stadium for various values of the applied magnetic field and by fitting the results using the matrix ensemble $H_{\mu\nu}$ of Eq. (2.1). More details of this procedure will be given later.

Next, we must consider the coupling of the closed stadium with two ideal external leads. The Hamiltonian operator for the total system (comprising stadium and leads) is given by

$$H = \sum_{\mu,\nu=1}^{N} |\mu\rangle\, H_{\mu\nu} \langle\nu| + \sum_{c=1}^{M} \int d\mu(E,c) |E,c\rangle\, E \langle E,c|$$

$$+ \sum_{\mu,c} \int d\mu(E,c) \{ |E,c\rangle\, W_{c\mu}(E) \langle\mu| + \text{h.c.} \}. \tag{2.4}$$

The first term represents the stadium Hamiltonian as discussed above. The second term describes the M free electron channels in both of the leads. Each channel c is defined by a propagating transverse mode with an energy E below the Fermi energy. Thus $|E,c\rangle$ is the plane wave mode of transverse energy ε_c and longitudinal momentum $\hbar k$, such that the total energy of the state is $E = \varepsilon_c + \hbar^2 k^2/(2m_c^*)$, with m_c^* being the effective mass of the electrons, and $d\mu(E,c)$ is the corresponding energy measure:

$$d\mu(E,c) = \sqrt{(m_c^*/8\pi^2\hbar^2)}\, \Theta(E - \varepsilon_c) \frac{dE}{\sqrt{E - \varepsilon_c}}, \tag{2.5}$$

where $\Theta(E)$ denotes the step function. The final term contains the lead–stadium couplings. We have denoted by $W_{c\mu}$ the matrix element which connects the basis state $|\mu\rangle$ in the stadium's internal region with channel c, where $\mu = 1, 2, ..., N$ and $c = 1, 2, ..., M$. We assume that the breaking of time-reversal symmetry by the magnetic field B need only be taken into account in the Hamiltonian (2.1) of the stadium and not in the matrix elements $W_{c\mu}$ describing the passage of the electron between one of the channels and the stadium interior. Accordingly, we set $W_{c\mu} = W^*_{c\mu}$.

General quantum scattering theory allows us to write down the scattering matrix S_{cd} for the problem at hand [13]. It has the form

$$S_{cd} = \delta_{cd} - 2i\pi \sum_{\mu,\nu} W_{c\mu}[D^{-1}]_{\mu\nu} W_{d\nu}, \tag{2.6}$$

where the inverse propagator $D_{\mu\nu}$ is given by

$$D_{\mu\nu} = E\delta_{\mu\nu} - H_{\mu\nu} + i\pi \sum_c W_{c\mu} W_{c\nu}. \tag{2.7}$$

Here, the energy E is taken to be the Fermi energy E_F, and thus will ultimately be set to zero. The scattering matrix defined in Eqs. (2.6), (2.7) is manifestly unitary but not symmetric (unless $t = 0$). In writing Eq. (2.7), we have assumed that the energy dependence of the matrix elements $W_{c\mu}$ is negligible over an energy interval of the order of d, and we have therefore omitted a principal-value integral. It was shown in Ref. [14] that taking such an integral along complicates the calculation but does not affect the GOE scattering result, except for a redefinition of the transmission coefficients introduced below. Insertion of Eqs. (2.6) and (2.7) into the Landauer formula (1.1) furnishes us with a formal expression for the conductance g in terms of the Hamiltonian, which must now undergo ensemble averaging. For this purpose, it is useful to note the compact representation

$$S_{ab} S^*_{ab} = 4 \operatorname{Tr} \Omega^a D^{-1} \Omega^b [D^{-1}]^\dagger, \tag{2.8}$$

where the trace "Tr" runs over the level index μ and we have introduced the quantities

$$\Omega^c_{\mu\nu} = \pi W_{c\mu} W_{c\nu}. \tag{2.9}$$

It is worth pointing out at this stage that the parameters $\{W_{c\mu}\}$ will not appear explicitly in our final expression for $\bar{g}$, for the following reason: The random-matrix ensemble (2.1) is invariant under unitary transformations. Therefore, only unitary invariants constructed from the $W_{c\mu}$'s can appear in averages like $\bar{g}$. The only such invariants are the bilinear forms

$$v^2_{cd} = N^{-1} \sum_\mu W_{c\mu} W_{d\mu}. \tag{2.10}$$

Because of the summations in Eq. (1.1), we may assume without loss of generality that

$$v_{cd}^2 = v_c^2 \delta_{cd} \tag{2.11}$$

is diagonal in the channel indices if c and d are channels in the *same* lead. In assuming that this relation holds in general, we disregard short classical trajectories connecting the two leads. It turns out [14] that the v_c's enter into the final expression for $\bar{g}$ only in the form of the dimensionless coefficients

$$T_c = \frac{4\lambda X_c}{(\lambda + X_c)^2}, \tag{2.12}$$

where $X_c = \pi N v_c^2$. Likewise, the parameter λ specifying the local level density appears *only* in the combination T_c. Obviously, we have $0 \leqslant T_c \leqslant 1$ for all c. The quantities T_c are interpreted as transmission coefficients which measure the strength of the coupling between the cavity and the leads. The absence of barriers between cavity and leads motivates us to put $T_c = 1$ for all c, thereby maximizing the coupling to the leads. Then, $\bar{g}$ depends only on B and on M. All dependence on λ and on the Fermi energy has disappeared, and we are left with a calculation that contains *no* free parameters.

We emphasize that in our approach, all channels are strictly equivalent. There is no distinction made between channels in lead one and those in lead two. Hence, the sums in Eq. (1.1) are restricted only in the sense that they must be carried out over different sets $\{a\} \neq \{b\}$ of channels. Clearly, this equivalence assumption is justified only for long delay times of the electron inside the billiard. The experiments [6–8] are arranged in such a way that it is unlikely for electrons to pass directly from one lead to the other, lending credence to this assumption.

A crucial constraint applies in our calculation of $\bar{g}$. We cannot use the usual perturbative expansion in powers of $1/M$ as done in previous work [1]. First, experiments typically have a small number of open channels, $M = 2, 4$, or 6. For example, in the experiments of Refs. [6, 7], the stadium billiard is connected to two almost one-dimensional contacts in each of which the number of channels $M/2$ depends on the gate voltage and varies between one and three. For $M \gg 1$, the semiclassical approximation is exptected to work. For small M, on the other hand, we enter a different regime which is not accessible to perturbative methods. Physically speaking, the levels in the billiard are broadened by the coupling to the leads. For small M, this broadening is of the order of the mean level spacing d. This is the regime we mainly address in this paper, although our results are generic and apply to all values of M. Second, and more importantly, we shall see that the dependence of the mean conductance on t enters in the combination t/M. This implies that the perturbative results will be good only as long as $t \ll M$, but this will be shown to fall short of the crossover region. Consequently, we must resort to a computation that is exact in the number of channels M. This we shall achieve by using the method of the supersymmetric generating function [15, 14] to find an expression for $\bar{g}(t, M)$ valid for all values of t and M.

3. Magnetic-Field Parameter

In order to completely specify our model in terms of physical quantities, it remains to discuss how one can relate t to the strength of the magnetic field B. A previous investigation of the GOE → GUE transition in terms of the same magnetic-field parameter t by Pandey and Mehta [16] offers the possibility of relating t and B by direct inspection of the spectral fluctuations of the stadium for different values of the external magnetic field. Also, in more recent work, Ozório de Almeida [17] derived semiclassically the dependence of t on B for a chaotic system in terms of purely classical correlators. Although these classical quantities are difficult to compute with very good accuracy, an efficient average over the phase space is sufficient to provide another independent quantitative estimate of t in terms of physical quantities. In this section, both approaches are discussed.

We shall start with the first method and consider the Schrödinger equation for an electron moving in a billiard in the presence of an external magnetic field B. Working in the gauge $\mathbf{A}(x, y) = (-By/2, Bx/2, 0)$ and introducing rescaled coordinates $x' = \eta^{1/2}x$, $y' = \eta^{1/2}y$ with $\eta = 4\pi/\mathscr{A}$, $\mathscr{A}$ being the area of the stadium, we arrive at the Schrödinger equation in the form

$$\left[\nabla^2 - \frac{i}{2}\frac{\phi}{\phi_0}\left(x'\frac{\partial}{\partial y'} - y'\frac{\partial}{\partial x'}\right) - \frac{\pi}{8}\left(\frac{\phi}{\phi_0}\right)^2 (x'^2 + y'^2) + \frac{\mathscr{A}}{4\pi}\frac{2m}{\hbar^2}E\right]\psi(x', y') = 0, \quad (3.1)$$

where $\phi_0 = h/e$ is the magnetic flux quantum. The boundary conditions are given by $\psi(x', y') = 0$ for $(x', y') \in \partial D$, where ∂D is the border of the stadium. This is the eigenvalue equation that we solve numerically. We subsequently drop the primes on the coordinates. Recalling the Weyl formula for the average level density ρ^{av} for Dirichlet boundary conditions in the absence of magnetic field,

$$\rho^{\text{av}}(E) = \frac{\mathscr{A}}{4\pi}\frac{2m}{\hbar^2} - \frac{\mathscr{P}}{4\pi}\sqrt{2m/\hbar^2 E} + \cdots, \quad (3.2)$$

where $\mathscr{P}$ denotes the perimeter of the stadium, we see that, up to a correction of order $E^{-1/2}$, the rescaling leads to an energy spectrum of average density one in units of $2m/\hbar$.

The linear term in ϕ in Eq. (3.1) breaks time-reversal symmetry. The term quadratic in ϕ brings the system to the integrable limit as one increases ϕ. This term is of little importance for the range of B fields relevant to the experiment under consideration. The structure of Eq. (3.1) already suggests a linear dependence between the parameter $\sqrt{t}$ and ϕ/ϕ_0, provided that the operator $\hat{O} = i(x\partial/\partial y - y\partial/\partial x)$ locally connects different states efficiently, thereby mimicking a random matrix. Although the latter assumption seems to be rather natural, in a dynamical system, the variance of the matrix elements associated with $\hat{O}$ will typically vary with the energy. A quantitative prediction of the trend is possible by invoking the following arguments: For $B = 0$, Berry conjectured [18] that the wavefunction of an eigenstate of energy $E = \hbar^2 k^2/2m$ for a classically chaotic billiard is typically given by a superposition of plane waves $a_i \exp(i\mathbf{k}_i \cdot \mathbf{r})$ with $\mathbf{k}_i$ randomly oriented, $k = |\mathbf{k}_i|$ and Gaussian amplitudes a_i. Since according to this conjecture, $\hat{O} = -\hbar(\mathbf{r} \times \hat{\mathbf{k}})_z$, it follows that for eigenstates μ and ν with $k_\nu \approx k_\mu \approx k$, $\hat{O} \approx -\hbar k \langle \nu | (\mathbf{r} \times \hat{\mathbf{k}})_z | \mu \rangle = k v_{\nu\mu}$, where $\hat{\mathbf{k}} = \mathbf{k}/k$ and $v_{\nu\mu}$ is a random variable. In order to compute the wavefunctions of the stadium with good accuracy, Heller [19] typically uses $k\mathcal{P}$ plane-wave components. Taking into consideration the number of components and the randomness of a_i, one can estimate that $\overline{v_{\mu\nu}^2} \propto k^{-1}$. From this naive argument, one concludes that $\hat{O}^2 \propto k$. Therefore, any numerical study of the GOE → GUE transition in a dynamical system will be quantitative within a given energy range. A precise correction for the energy dependence of our results can be made via the semiclassical approach and will be discussed later.

Analytical results for the GOE → GUE transition are available for the fluctuating part of the level–level density correlator, defined as

$$Y_2(\varepsilon/d) = d^2 \overline{\rho^{\text{fl}}(E)\, \rho^{\text{fl}}(E+\varepsilon)}, \tag{3.3}$$

where ρ is the mean level density. For the model defined in Eq. (2.1), one finds in Ref. [16] the following analytical expression for Y_2 as a function of $x = \varepsilon/d$ and t:

$$Y_2(x,t) = \left(\frac{\sin \pi x}{\pi x}\right)^2 - \frac{1}{\pi^2} \int_0^\pi dr\, r e^{2tr^2/\pi^2} \sin rx \int_\pi^\infty dq\, \frac{\sin qx}{q}\, e^{-2tq^2/\pi^2}. \tag{3.4}$$

Unfortunately, this statistical measure is not very efficient for quantifying t in practice. Many numerical studies show that a more adequate one is the $\Delta_3(x)$ statistic, which can be related to $Y_2(x)$ straightforwardly [20]. We have proceeded as follows: For different values of ϕ/ϕ_0, we computed the Δ_3 level statistics and adjusted t in Eq. (3.4) to obtain the best fit. The results for the quarter stadium, with radius $R = 1$ and length* $l = 0.5$ are presented in Figs 3a–c. The statistical analysis of the spectra was performed over an energy interval where the level density stays almost constant. (Only the leading term in Eq. (3.2) is relevant.) Since we are interested in a quantitative determination of the transition parameter, care must be taken with respect to the following: (a) Berry's saturation of the Δ-statistics, which would jeopardize the method. By restricting our analysis to $x < 10$, this problem is circumvented. (b) The lowest eigenstates of the stadium show some marked regularities. In our analysis, we considered levels only above the fiftieth. In so doing, the agreement for $\phi = 0$ with the GOE results (for levels up to 300) is very good. (c) Spectral modulations due to "bouncing ball" orbits would make comparison of the model (i.e., Eq. (2.1)) with the billiard difficult. For our choice of a relatively small l/R, similar to the experimental situation, there are no sizeable modulations. The numerical results indicate that $\sqrt{t} = 3.87\phi/\phi_0$. The uncertainty in the fitting of the best $\Delta_3(x,t)$ is about 5% due to statistical error bars. Here, we determined t for a system containing approximately $n = 400$ electrons. The

* In this section, the length of the stadium is denoted by l.

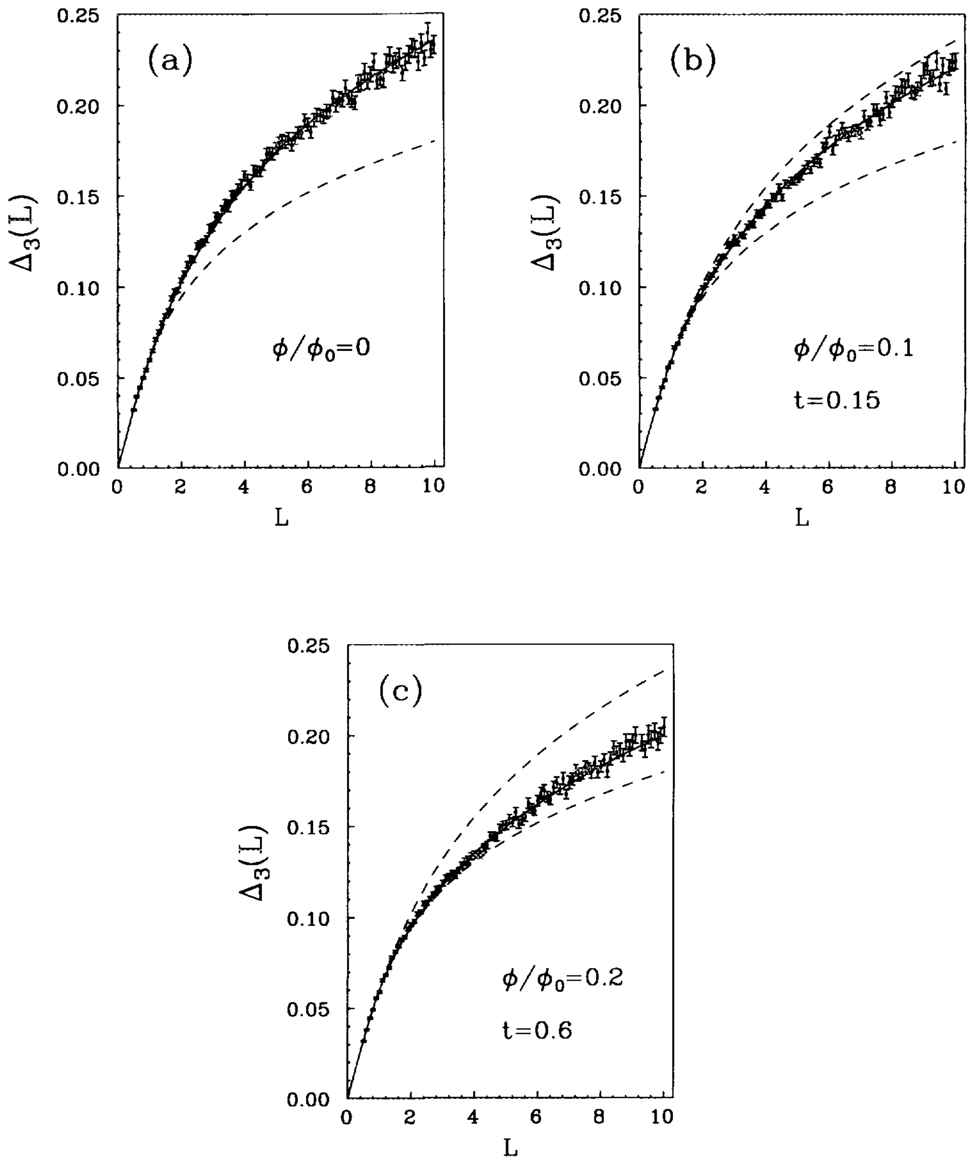

FIG. 3. (a) Graph of the Δ_3 statistic versus energy interval L for flux angle $\phi/\phi_0=0$. The solid line corresponds to setting $t=0$ in Eq. (3.4) (i.e., the GOE result). The dashed line is the GUE result. (b) Graph of the Δ_3 statistic versus energy interval L for flux angle $\phi/\phi_0=0.1$. The solid line shows the best fit to the data points furnished by Eq. (3.4), obtained by adjusting $t=0.15$ The upper and lower dashed lines correspond to the GOE and GUE results, respectively. (c) Graph of the Δ_3 statistic versus energy interval L for flux angle $\phi/\phi_0=0.2$. The solid line shows the best fit to the data points furnished by Eq. (3.4), obtained by adjusting $t=0.6$ The upper and lower dashed lines correspond to the GOE and GUE results, respectively.

physical situation of interest is more likely to deal with $n \approx O(10^3)$, and the question is how our result will be changed for this region of the spectrum.

A partial answer to this question was given by Berry and Robnik [21], who analysed the semiclassical transition parameter for a billiard threaded by an Aharonov–Bohm flux line. They found that for a fixed ϕ/ϕ_0, t increases as $n^{1/2}$. A complete semiclassical treatment can be found in Ref. [17], where t is expressed in terms of the density of states and purely classical quantities:

$$\sqrt{t} = \frac{K(E)\, \rho^{\text{av}}(E)}{4h}, \tag{3.5}$$

where $K(E)$ is a classical quantity which measures the average magnetic flux squared that is enclosed by a typical trajectory. We should like to stress that it is not the purpose of this discussion to further improve the semiclassical approach, but rather to apply the findings of Ref. [17] to obtain some quantitative results. For this purpose, we shall present the semiclassical results in a nutshell, motivate their origins, discuss briefly $K(E)$, and rescale it in a convenient way. Finally we compute $K(E)$ for a certain geometry and determine the t-dependence in a way independent from the previous one.

After some manipulations on the Gutzwiller trace formula, Ref. [17] gives $Y_2(\varepsilon)$ as

$$Y_2(\varepsilon) = \left(\frac{d}{2\pi h}\right)^2 \int_{-\infty}^{\infty} d\tau\, |\tau|\, e^{(i\varepsilon\tau - 2\eta|\tau|)/h^2} (1 + \langle e^{-\langle \Delta S^2(\tau) \rangle_{\text{p.s.}}/h^2} \rangle_E), \tag{3.6}$$

where the average $\langle \cdots \rangle_E$ should be taken over an energy interval that is large compared with ε and the smoothing parameter η should be small compared with ε. The central dynamical quantity to be computed is the phase space average $\langle \cdots \rangle_{\text{p.s.}}$ of ΔS^2 as a function of time τ, which replaces the usual average over periodic orbits. It is interesting to note that all the physics due to the presence of the magnetic field is contained in the latter quantity. Introducing the dynamical variable $\tilde{\varphi}(\tau) = \mathbf{A}(\mathbf{r}) \cdot \dot{\mathbf{r}} = eB\varphi(\tau)$, one can write

$$\langle \Delta S^2(\tau) \rangle_{\text{p.s.}} = 8(eB)^2 \int_0^\tau dt \int_0^{\tau - t} dt' \, \langle \varphi(t)\, \varphi(t + t') \rangle_{\text{p.s.}}. \tag{3.7}$$

There is considerable work in the literature dealing with long power law tails $t' \to \infty$ of $\langle \varphi(t)\, \varphi(t + t') \rangle_{\text{p.s.}}$ and similar correlators. These works and our numerical studies assure us that $\langle \varphi(t)\, \varphi(t + t') \rangle_{\text{p.s.}}$ decays sufficiently fast so that one can define

$$K(E, \tau) = \int_0^\tau dt' \, \langle \varphi(0)\, \varphi(t') \rangle_{\text{p.s.}} \tag{3.8}$$

and $\langle \Delta S^2(\tau) \rangle_{\text{p.s.}} \approx 8(eB)^2 K(E, \tau \to \infty)\, \tau$ for sufficiently large τ. The quantity $K(E)$ has trivial E and $\mathscr{A}$ dependence, and it is convenient to single out these variables so as to obtain a correlator κ defined for a unit area and unit velocity, which depends solely on the shape of the billiard. This can be easily done by a proper rescaling of the coordinates and the time; as a result, one has

$$K(E, \tau) = \sqrt{(2E\mathscr{A}^3/m)}\,\kappa(\tau). \tag{3.9}$$

Inserting this into the semiclassical expression for the transition parameter and recalling Eq. (3.2), one arrives at the relation

$$t = \left(\frac{\phi}{\phi_0}\right)^2 (N^{\mathrm{av}}(E))^{1/2}\,\pi^{5/2}\kappa(\tau \to \infty), \tag{3.10}$$

where $N^{\mathrm{av}}(E)$ is the cumulative level density, which counts the number of single-particle states in the billiard up to the energy E. This result gives us the correct energy scaling of the problem, implying that for a fixed magnetic field the GOE → GUE transition will become faster as the number of electrons n increases (in agreement with Ref. [21] and our naive estimate). Moreover, it indices that $\sqrt{t}$ depends linearly on ϕ/ϕ_0 as a naive inspection of Eq. (3.1) would suggest. Equation (3.10) also allows one to obtain quantitative results by determining κ for the specific system under consideration. In Fig. 4, we present our result for κ, obtained by a numerical average over the phase space. This was done by computing 10^7 trajectories with randomly chosen initial points. (It is of interest to mention that such an average has to be performed over closed paths; otherwise the results depend on where the origin of the coordinate system is taken.) Our result is $\kappa(\tau \to \infty) \simeq 0.06$. It follows that $\sqrt{t} = \alpha[N^{\mathrm{av}}(E)]^{1/4}\,\phi/\phi_0$, with $\alpha \simeq 1.05$ in good agreement with $1.03 = 3.87/(200)^{1/4}$ obtained by the other method (where we should note that $N^{\mathrm{av}}(E)$, being the number of levels, is half as small as n due to spin degeneracy).

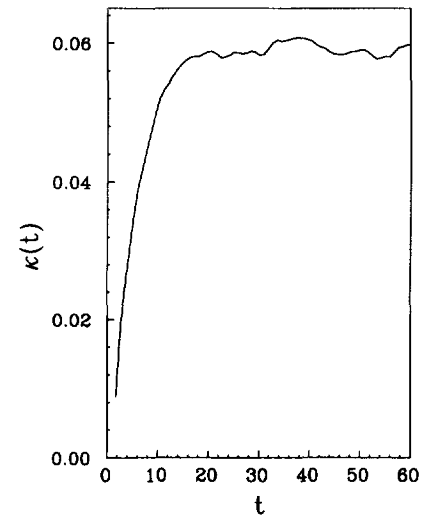

FIG. 4. The correlator $\kappa(t)$ (defined in Eq. (3.9)) as a function of the time t computed as a numerical average over phase space, taking 10^7 trajectories. The $t \to \infty$ asymptote is $\kappa \simeq 0.06$.

4. Supersymmetry Formalism

To calculate the ensemble average $\bar{g}$, we use Efetov's supersymmetry approach in the version described in Ref. [14]. In that paper, the scattering problem formulated in Section 2 was worked out *without* the GOE-symmetry-breaking term in Eq. (2.1). The present calculation follows very closely the lines of Ref. [14].

The appearance of the symmetry-breaking term causes the scattering matrix not to be symmetric. This leads to a set of source terms not considered in Ref. [14]. Moreover, the appearance of the symmetry-breaking term in the effective Lagrangian of the zero-dimensional non-linear sigma model derived in the $N \to \infty$ limit makes it much harder to calculate the terms of highest (eighth) order in the anticommuting integration variables. These are the terms that yield the answer to our problem. (All terms of lower order yield vanishing contributions. This statement, well known in the absence of symmetry-breaking, applies also in the case considered here [22].) There are two ways to overcome this difficulty: (i) One may use a parametrization of the Q-matrix developed by Altland, Efetov, and Iida [23] which is particularly tailored to time-reversal symmetry breaking, or (ii) one may use Efetov's original parametrization [15] of the Q-matrix in conjunction with the method developed in Ref. [5] to simplify the form of the symmetry-breaking term.

We have tried both approaches. It turns out that method (ii) is very much simpler: The terms of highest order can be obtained by hand, i.e., without recourse to a computer-based algorithm. The reason is that in comparison with the level correlation function, the source terms needed to obtain the elements of the scattering matrix depend on Q in a more complicated way: The matrix Q appears in the denominator. In such a case, the method of Ref. [23] becomes very unwieldy. This is why we have adopted method (ii).

In what follows, we have attempted to render a reasonably complete and systematic account of the derivation of the explicit formal expression for $\bar{g}$ as given within the supersymmetry formalism. Although much of this material has appeared previously in various guises, these sources are rather fragmentary, and so we consider it worthwhile to present here a pedagogical treatment that is hopefully comprehensible to the relatively uninitiated reader who has a little background in Grassmann algebra and general random-matrix theory. At the same time, we aim to achieve consistency of notation and conventions. On the other hand, the expert reader should simply take note of our generating function as given by Eqs. (4.14)–(4.16), which incorporates the new source matrix that is employed in the present calculation. This is given in Eq. (4.19). From here, one can proceed directly to the final integral expression over supermatrices for $\overline{|S_{ab}|^2}$, which is presented in Eq. (4.42).

4.1. *Generating Function*

Let us consider a generating function given by

$$\begin{aligned} Z(\varepsilon) &= \operatorname{detg}^{-1}[D + J(\varepsilon)] \\ &= \exp\{-\operatorname{Trg} \ln[D + J(\varepsilon)]\}, \end{aligned} \tag{4.1}$$

where the symbols "Detg" and "Trg" denote the graded determinant and graded trace as defined in the supersymmetric formalism of Ref. [14]. Here, the inverse propagator D has been extended to a $4N \times 4N$ supermatrix,

$$D = E - H) \mathbf{1}_4 + \tfrac{1}{2}\omega L + i\Omega L, \tag{4.2}$$

where $\Omega = \sum_c \Omega^c$ and $L^{\alpha\alpha'}_{pp'} = (-1)^{p+1} \delta_{pp'} \delta^{\alpha\alpha'}$ is the diagonal supermatrix that distinguishes between advanced and retarded parts of D. We have $D_{pp'} = [\mathrm{diag}(D, D^\dagger)]_{pp'}$, where $p, p' = 1, 2$, with $p = 1$ referring to the retarded block and $p = 2$ to the advanced block. The indices $\alpha, \alpha' = 0, 1$ determine the grading (with $\alpha = 0$ for the commuting (bosonic)components and $\alpha = 1$ for the anticommuting (fermionic) components). We have also allowed for the presence of a difference $\omega = E_1 - E_2$ between the energy arguments of the two S-matrix elements that multiply together to form g. According to Eq. (4.2), D_{11} corresponds to energy $E_1 = E + \frac{1}{2}\omega$ while D_{22} corresponds to energy $E_2 = E - \frac{1}{2}\omega$. We shall ultimately set $\omega \to 0$, but keep it in the interim to facilitate comparison with other related work [23].

The source supermatrix $J(\varepsilon)$ depends on a set of parameters ε_m labelled by some (multi)-index m and is taken to have the general form

$$J(\varepsilon) = \varepsilon_m M_m \tag{4.3}$$

for some set of $4N \times 4N$ supermatrices $M_m(\mu, \nu)$. It then follows that

$$\left.\frac{\partial^2}{\partial \varepsilon_m \partial \varepsilon_n} Z(\varepsilon)\right|_{\varepsilon=0} = \mathrm{Trg}\, M_m D^{-1} \cdot \mathrm{Trg}\, M_n D^{-1} + \mathrm{Trg}\, M_m D^{-1} M_n D^{-1}. \tag{4.4}$$

For the problem at hand, we shall make the choice of source matrix

$$J_{\mu\nu}(\{\varepsilon^j_{ab}\}) = \pi \sum_{a,b} \sum_{j=1}^{2} I(j)\, W_{a\nu} \varepsilon^j_{ab} W_{b\mu}, \tag{4.5}$$

where

$$I^{\alpha\alpha'}_{pp'}(j) = -k^{\alpha\alpha'} \delta_{pp'} \delta_{pj} \tag{4.6}$$

is a projector onto the $p = j$ block with $k^{\alpha\alpha'} = (-1)^\alpha \delta^{\alpha\alpha'}$. In this case, the second term on the RHS of Eq. (4.4) vanishes if we differentiate with respect to $\varepsilon_m = \varepsilon^1_{ab}$ and $\varepsilon_n = \varepsilon^2_{ba}$. Comparison with Eq. (2.8) then leads to the identification

$$|S_{ab}|^2 = \left.\frac{\partial^2}{\partial \varepsilon^1_{ab} \partial \varepsilon^2_{ba}} Z(\varepsilon)\right|_{\varepsilon=0}. \tag{4.7}$$

4.2. *Ensemble Average*

As a Gaussian superintegral, the generating function $Z(\varepsilon)$ reads

$$Z(\varepsilon) = \int \mathscr{D}\varphi\, \mathscr{D}\bar{\varphi} \exp\left\{ i \sum_{p,\alpha} \langle \bar{\varphi}^\alpha_p, [(D + J)\varphi]^\alpha_p \rangle \right\}, \tag{4.8}$$

where we employ the notation

$$\langle F(\mu), G(\mu) \rangle \equiv \sum_{\mu} F(\mu)\, G(\mu), \tag{4.9}$$

and $\varphi_p^{\alpha}(\mu)$ is a four-component supervector field. The adjoint supervector is defined by $\bar{\varphi} = \varphi^{\dagger} s$ with

$$s_{pp'}^{\alpha\alpha'} = s_p^{\alpha} \delta^{\alpha\alpha'} \delta_{pp'}, \qquad s_p^{\alpha} = (-1)^{(1-\alpha)(1+p)}. \tag{4.10}$$

The presence of the supermatrix s in the definition of the adjoint ensures the convergence of the final integral representation of the ensemble-averaged generating function as a supermatrix non-linear σ-model and ensures the correct combination of compact and non-compact symmetries therein [24].

With the definitions

$$\Phi = \begin{pmatrix} \varphi \\ s\varphi^* \end{pmatrix}, \qquad \bar{\Phi} = \Phi^{\dagger} s, \tag{4.11}$$

we have the property

$$\bar{\varphi} A \varphi = \tfrac{1}{2}\, \bar{\Phi} \mathbf{A} \Phi, \tag{4.12}$$

where

$$\mathbf{A} = \begin{pmatrix} A & 0 \\ 0 & A^{\mathrm{T}} \end{pmatrix} \tag{4.13}$$

for *any* supermatrix A that is diagonal in the graded indices α, α'. The eight-dimensional supervector Φ possesses the "charge-conjugation" or reality property $\Phi^* = C\Phi$ which can equivalently be expressed as a condition of Majorana type, viz. $\bar{\Phi} = (\Lambda\Phi)^{\mathrm{T}}$, where the matrices C, Λ are defined in Appendix A. In view of Eq. (4.12), Eq. (4.8) may be cast in a form that renders the GOE ensemble average particularly simple, namely,

$$Z(\varepsilon) = \int \mathscr{D}\Phi \exp\left\{ \frac{i}{2} \sum_{p,\alpha,r} \langle \bar{\Phi}_{pr}^{\alpha}, [(\mathbf{D} + \mathbf{J})\, \Phi]_{pr}^{\alpha} \rangle \right\}. \tag{4.14}$$

The indices r, r' span the additional matrix structure due to the doubling of dimensionality implied by Eq. (4.11). We shall say that the index r labels the "GOE blocks." Here, we have

$$\mathbf{D} = E\mathbf{1}_8 - \mathbf{H} + \tfrac{1}{2}\omega L + i\Omega L, \tag{4.15}$$

and the definitions of $\mathbf{H}$ and $\mathbf{J}$ are those implied by Eq. (4.13). In particular,

$$\mathbf{H} = \begin{pmatrix} H & 0 \\ 0 & H^{\mathrm{T}} \end{pmatrix} = H^{(S)} \otimes \mathbf{1}_8 + i\sqrt{(t/N)}\, H^{(A)} \otimes \tau^3, \tag{4.16}$$

where the matrix τ^3 is defined in Appendix A. We should note the following symmetry properties: (i) transformations $T\colon \Phi \mapsto T\Phi$ that preserve the relation $\Phi^* = C\Phi$ obey

$$T^* = CTC^{-1} \tag{4.17}$$

and (ii) transformations $T\colon \Phi \mapsto T\Phi$ that leave the bilinear $\bar{\Phi}\Phi$ invariant are those which satisfy

$$T^{-1} = sT^{\dagger}s. \tag{4.18}$$

The set of 8×8 supermatrices respecting conditions (4.17) *and* (4.18) forms a supergroup with compact and non-compact bosonic subgroups. In the absence of symmetry-breaking and source terms, the generating function in Eq. (4.14) is invariant under this supergroup.

Special attention should be paid to the induced r-space structure of the source supermatrix $\mathbf{J}$,

$$\mathbf{J}_{\mu\nu} = \pi \sum_{a,b} \sum_{j=1}^{2} \mathbf{I}_1(j)\, W_{a\nu} \varepsilon_{ab}^{(s)j} W_{b\mu} + \pi \sum_{a,b} \sum_{j=1}^{2} \mathbf{I}_2(j)\, W_{a\nu} \varepsilon_{ab}^{(a)j} W_{b\mu}, \tag{4.19}$$

where $\varepsilon_{ab}^{(s)j}$ and $\varepsilon_{ab}^{(a)j}$ are the parts of ε_{ab}^{j} symmetric and antisymmetric in the indices a, b, respectively, and we have introduced $\mathbf{I}_1(j) = I(j) \otimes \mathbf{1}_2$, $\mathbf{I}_2(j) = I(j) \otimes \tau^3$. Consequently, the symmetrized form of $|S_{ab}|^2$ can be written as the sum of two terms:

$$\frac{1}{2}\{|S_{ab}|^2 + |S_{ba}|^2\} = \frac{1}{4} \frac{\partial^2}{\partial \varepsilon_{ab}^{(s)1} \partial \varepsilon_{ba}^{(s)2}} Z(\varepsilon^{(s)}, 0) \Big|_{\varepsilon=0} + \frac{1}{4} \frac{\partial^2}{\partial \varepsilon_{ab}^{(a)1} \partial \varepsilon_{ba}^{(a)2}} Z(0, \varepsilon^{(a)}) \Big|_{\varepsilon=0} \tag{4.20}$$

for $a \neq b$. Cast in this form, the contribution to $|S_{ab}|^2$ from the first term on the RHS of the equation above is generated by essentially the same source matrix as used in Ref. [14] for the pure GOE problem; the second term is generated by a new source matrix. This latter term must vanish for $t = 0$ and accounts for the asymmetry of the S-matrix.

The factor in the integral of Eq. (4.14) which undergoes ensemble averaging can be decomposed as

$$\overline{\exp\left\{-\frac{i}{2} \sum_{p,\alpha,r} \langle \bar{\Phi}_{pr}^{\alpha}, (\mathbf{H}\Phi)_{pr}^{\alpha} \rangle\right\}}^{\mathbf{H}} = \overline{\exp\{-i \operatorname{Tr}_{\mu} H^{(S)} \mathscr{S}^{(0)}\}}^{S} \cdot \overline{\exp\{\sqrt{t/N} \operatorname{Tr}_{\mu} H^{(A)} \mathscr{S}^{(3)}\}}^{A}, \tag{4.21}$$

where we have introduced the ordinary $N \times N$ matrices in the level indices μ, ν,

$$\mathcal{S}^{(j)}(\nu, \mu) = \tfrac{1}{2} \sum_{\substack{p, \alpha \\ r, r'}} \bar{\Phi}^{\alpha}_{pr}(\mu)\, \tau^{j}_{rr'}\, \Phi^{\alpha}_{pr'}(\nu), \tag{4.22}$$

and we take $\tau^0 = \mathbf{1}$. Gaussian ensemble averaging leads to the relations

$$\begin{aligned} \overline{\exp\{-i \operatorname{Tr}_{\mu} H^{(S)} \mathcal{S}^{(0)}\}}^{\mathrm{S}} &= \exp\left\{-\frac{\lambda^2}{4N} \operatorname{trg} S^2\right\}, \\ \overline{\exp\{\sqrt{t/N} \operatorname{Tr}_{\mu} H^{(A)} \mathcal{S}^{(3)}\}}^{\mathrm{A}} &= \exp\left\{-\frac{t}{N} \frac{\lambda^2}{4N} \operatorname{trg} \tau^3 S \tau^3 S\right\}, \end{aligned} \tag{4.23}$$

where

$$S^{\alpha, \alpha'}_{pr, p'r'} \equiv \sum_{\mu} \Phi^{\alpha}_{pr}(\mu)\, \bar{\Phi}^{\alpha'}_{p'r'}(\mu) \tag{4.24}$$

is a supermatrix. These results are easily derived with the aid of the relations

$$\begin{aligned} \sum_{\mu, \nu} \mathcal{S}^{(0)}(\nu, \mu)\, \mathcal{S}^{(0)}(\mu, \nu) &= \tfrac{1}{4} \operatorname{trg} S^2, \\ \sum_{\mu, \nu} \mathcal{S}^{(0)}(\nu, \mu)\, \mathcal{S}^{(0)}(\nu, \mu) &= \tfrac{1}{4} \operatorname{trg} S^2, \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} \sum_{\mu, \nu} \mathcal{S}^{(3)}(\nu, \mu)\, \mathcal{S}^{(3)}(\mu, \nu) &= +\tfrac{1}{4} \operatorname{trg} \tau^3 S \tau^3 S, \\ \sum_{\mu, \nu} \mathcal{S}^{(3)}(\nu, \mu)\, \mathcal{S}^{(3)}(\nu, \mu) &= -\tfrac{1}{4} \operatorname{trg} \tau^3 S \tau^3 S. \end{aligned} \tag{4.26}$$

The ensemble-averaged generating function then becomes

$$\overline{Z(\varepsilon)} = \int \mathcal{D}\Phi\, e^{i\mathcal{L}^{\#}(S)} \exp\left\{\frac{i}{2} \sum_{p, \alpha, r} \langle \bar{\Phi}^{\alpha}_{pr}, [(E\mathbf{1} + \tfrac{1}{2}\omega L + i\Omega L + \mathbf{J})\, \Phi]^{\alpha}_{pr} \rangle\right\}, \tag{4.27}$$

where

$$i\mathcal{L}^{\#}(S) = -\frac{\lambda^2}{4N}\left[\operatorname{trg} S^2 + \frac{t}{N} \operatorname{trg} \tau^3 S \tau^3 S\right]. \tag{4.28}$$

4.3. *Hubbard–Stratonovich Transformation*

Following Ref. [25], let us introduce unity under the Φ-integral in the form

$$1 = \frac{1}{\mathcal{N}(S)} \int \mathcal{D}\sigma\, e^{iW(\sigma, S)}, \qquad \mathcal{N}(S) = \int \mathcal{D}\sigma\, e^{iW(\sigma, S)}, \tag{4.29}$$

followed with an interchange of the σ and Φ integrations, where σ spans some appropriate space of supermatrices, not yet specified. The quantity $W(\sigma, S)$ can be any function provided the integral in Eq. (4.29) is convergent. The choice

$$W(\sigma, S) = -\mathcal{L}^{\#}\left(S - \frac{iN}{\lambda}\sigma\right) \tag{4.30}$$

leads to

$$\begin{aligned} i\mathcal{L}^{\#}(S) + iW(\sigma, S) = &-\frac{N}{4}\left[\operatorname{trg}\sigma^2 + \frac{t}{N}\operatorname{trg}\tau^3\sigma\tau^3\sigma\right] \\ &-\frac{i\lambda}{2}\sum_{p,\alpha,r}\left\langle \bar{\Phi}^{\alpha}_{pr}, \left[\left(\Sigma + \frac{t}{N}\tau^3\Sigma\tau^3\right)\Phi\right]^{\alpha}_{pr}\right\rangle, \end{aligned} \tag{4.31}$$

where $\Sigma_{\mu\nu} = \sigma\delta_{\mu\nu}$. The quartic dependence on Φ in the exponent of the generating function implied in $\mathcal{L}^{\#}(S)$ is thus eliminated, and if the space of σ's is chosen such that $\mathcal{N}(S) = \mathcal{N}(0)$ (i.e., shifts are allowed), then the remaining quadratic dependence on Φ is amenable to exact integration, which yields

$$\begin{aligned} \overline{Z(\varepsilon)} = &\int \mathcal{D}\sigma \exp\left\{-\frac{N}{4}\left[\operatorname{trg}\sigma^2 + \frac{t}{N}\operatorname{trg}\tau^3\sigma\tau^3\sigma\right]\right\} \\ &\times \exp\left\{-\frac{1}{2}\operatorname{Trg}_{\mu}\ln\left[E\mathbf{1} + \frac{1}{2}\omega L + i\Omega L + \mathbf{J} - \lambda\left(\Sigma + \frac{t}{N}\tau^3\Sigma\tau^3\right)\right]\right\}. \end{aligned} \tag{4.32}$$

The appropriate space of matrices σ is given by those which assume the form

$$\sigma = T^{-1}(P - i\eta L)\,T, \tag{4.33}$$

where the P, which represent massive modes, are essentially Hermitian and block-diagonal in the p-index, and the T span the set of matrices defined by Eqs. (4.17) and Eq. (4.18) modulo the subgroup that commutes with L. The positive constant η is required for convergence, and its precise value will be determined from the saddle-point equation. Next, we write $P - i\eta L = \sigma_0 + \delta P$, where σ_0 is the unique diagonal saddle-point that lies within the integration manifold, and δP are the fluctuations around it. Thus we have $\sigma = \sigma_G + T^{-1}\delta P T$ with $\sigma_G = T^{-1}\sigma_0 T$. The massive fluctuations can be integrated out in the limit $N \to \infty$, which leads to an expression $\overline{Z(\varepsilon)}$ identical with Eq. (4.32), except for the replacement $\sigma \to \sigma_G$ everywhere.

With $E = 0$ and neglecting ω (which is assumed to be of order $O(N^{-1})$), the saddle-point equation reads $\sigma_0 + \sigma_0^{-1} = 0$. Setting $\sigma_0 = x\mathbf{1} + yL$ leads to $x = 0$, $y = \pm i$. According to Eq. (4.33), the correct choice is $\sigma_0 = -iL$. So we write $\sigma_G = -iQ$, where $Q \equiv T^{-1}LT$. Having set $E = 0$, we can write

$$\overline{Z(\varepsilon)} = \int \mathcal{D}Q\, e^{i\mathcal{L}_{\text{eff}}(Q)} e^{i\mathcal{L}_{\text{source}}(Q;J)}, \tag{4.34}$$

where

$$i\mathscr{L}_{\rm eff}(Q) = \frac{t}{4}\,\mathrm{trg}\,\tau^3 Q\tau^3 Q - \frac{1}{2}\,\underset{\mu}{\mathrm{Trg}}\ln\left(\frac{1}{2}\omega L + i\Omega L + i\lambda\tilde{Q}\right) \tag{4.35}$$

and

$$i\mathscr{L}_{\rm source}(Q; J) = -\frac{1}{2}\,\underset{\mu}{\mathrm{Trg}}\ln\left[\mathbf{1} + \frac{1}{(1/2)\,\omega L + i\Omega L + i\lambda\tilde{Q}}\,\mathbf{J}\right], \tag{4.36}$$

with

$$\tilde{Q} \equiv Q + \frac{t}{N}\,\tau^3 Q \tau^3. \tag{4.37}$$

The summation over the level index μ (implied in the trace "Trg") can be performed by virtue of the orthogonality relation of Eqs. (2.10), (2.11), from which one can derive the result

$$\underset{\mu}{\mathrm{Tr}}\,(\bar{\omega} + \Omega)^n = (N - M)\,\bar{\omega}^n + \sum_c (\bar{\omega} + X_c)^n, \tag{4.38}$$

for $n = 1, 2, \ldots$ and any constant $\bar{\omega}$. (In our case, $\bar{\omega} = -i\omega/2$.) Consequently, $\mathscr{L}_{\rm eff}(Q)$ can be decomposed according to $\mathscr{L}_{\rm eff}(Q) = \mathscr{L}_{\rm free}(Q) + \mathscr{L}_{\rm ch}(Q)$ with

$$\begin{aligned} i\mathscr{L}_{\rm free}(Q) &= \frac{t}{4}\,\mathrm{trg}\,\tau^3 Q \tau^3 Q - \frac{N}{2}\,\mathrm{trg}\ln\left(\frac{1}{2}\omega L + i\lambda\tilde{Q}\right), \\ i\mathscr{L}_{\rm ch}(Q) &= -\frac{1}{2}\sum_c \mathrm{trg}\ln\left[1 + \frac{1}{(1/2)\,\omega L + i\lambda\tilde{Q}}\,iX_c L\right]. \end{aligned} \tag{4.39}$$

Next, we expand $\mathscr{L}_{\rm free}(Q)$ and $\mathscr{L}_{\rm ch}(Q)$ in the small quantities ω and t/N to leading order. Using the fact that $Q^2 = \mathbf{1}$, this yields

$$\begin{aligned} i\mathscr{L}_{\rm free}(Q) &= -\frac{t}{4}\,\mathrm{trg}\,\tau^3 Q \tau^3 Q + \frac{iN\omega}{4\lambda}\,\mathrm{trg}\,LQ, \\ i\mathscr{L}_{\rm ch}(Q) &= -\frac{1}{2}\sum_c \mathrm{trg}\ln\left(\mathbf{1} + \frac{X_c}{\lambda}LQ\right). \end{aligned} \tag{4.40}$$

We note that $i\mathscr{L}_{\rm free}(Q)$ coincides with the analogous quantity derived in Ref. [23]. At this stage, we implement the assumption of equivalent channels with ideal couplings, as already discussed in Section 2, by setting $X_c = \lambda$ for all c. Thus we obtain

$$i\mathscr{L}_{\rm ch}(Q) = -\frac{M}{2}\,\mathrm{trg}\ln(\mathbf{1} + LQ), \tag{4.41}$$

noting that $\sum_c 1 = M$ by definition. The source term can be treated similarly.

Letting $\omega \to 0$, we finally arrive at the expression

$$\overline{|S_{ab}|^2} = \int \mathscr{D}Q \exp\left[-\frac{M}{2} \operatorname{trg} \ln(\mathbf{1} + QL) - \frac{t}{4} \operatorname{trg} Q\tau^3 Q\tau^3 \right] \times \sum_{j=1}^{2} \operatorname{trg}\left[\frac{1}{\mathbf{1}+QL} \mathbf{I}_j(1) \frac{1}{\mathbf{1}+QL} \mathbf{I}_j(2) \right]. \tag{4.42}$$

The space of the supermatrices Q can be parametrized by 16 real variables. These can be chosen to comprise two "azimuthal" angles, three "Cayley–Klein" parameters, eight anticommuting variables, and three eigenvalues (1 compact and 2 non-compact). All but the three eigenvalue parameters can be integrated out analytically. The final threefold integral giving the exact result for $\overline{|S_{ab}|^2}$ in its maximally reduced analytical form must be further evaluated numerically. Our result in Eq. (4.42) differs from that of Ref. [14] by the appearance of the new source term containing $\mathbf{I}_2(p)$ and the symmetry-breaking term involving τ^3 in the exponent.

5. Exact Results

Using the method developed in Ref. [5], we can perform in Eq. (4.42) the integration over all the anticommuting, as well as over several of the commuting variables used in the parametrization of Q. We are then left with a threefold integration over real variables (the "eigenvalues" in Efetov's parametrization). A summary of the technical details involved in the derivation is presented in Appendix B.

For ease of subsequent discussion, it is convenient to separate the exact result for $\overline{|S_{ab}|^2}$ into two contributions:

$$\overline{|S_{ab}|^2} = \overline{|S_1|^2} + \overline{|S_2|^2}, \tag{5.1}$$

where $\overline{|S_1|^2}$ represents the part arising from the first source term ($j = 1$) of Eq. (4.42) and $\overline{|S_2|^2}$ that from the second source term ($j = 2$). Then we have, for $a \neq b$,

$$\overline{|S_1|^2} = \frac{1}{2} \int_{-1}^{+1} d\lambda \int_1^{\infty} d\lambda_1 \int_1^{\infty} d\lambda_2 \left(\frac{1+\lambda}{\lambda_1 + \lambda_2} \right)^M e^{-2t(\lambda_2^2 - 1)} \left[\frac{1-\lambda}{1+\lambda} + \frac{\lambda_1^2 + \lambda_2^2 - 2}{(\lambda_1 + \lambda_2)^2} \right] \times \left\{ \frac{1}{\mathscr{R}^2} [(1-\lambda^2)(1 + e^{-2t(1-\lambda^2)}) - (\lambda_1^2 - \lambda_2^2)(1 - e^{-2t(1-\lambda^2)})] + \frac{4t}{\mathscr{R}} [(1-\lambda^2) e^{-2t(1-\lambda^2)} + \lambda_2^2 (1 - e^{-2t(1-\lambda^2)})] \right\}, \tag{5.2}$$

where

$$\mathscr{R} = \lambda^2 + \lambda_1^2 + \lambda_2^2 - 2\lambda\lambda_1\lambda_2 - 1, \tag{5.3}$$

and

$$\overline{|S_2|^2} = \frac{1}{2}\int_{-1}^{+1} d\lambda \int_1^{\infty} d\lambda_1 \int_1^{\infty} d\lambda_2 \left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^M e^{-2t(\lambda_2^2-1)} \frac{1}{\mathscr{R}}$$
$$\times \left\{ -\left(1 + e^{-2t(1-\lambda^2)} - \frac{1-e^{-2t(1-\lambda^2)}}{t(1-\lambda^2)}\right) \frac{1-\lambda}{(1+\lambda)(\lambda_1+\lambda_2)} \right.$$
$$\times \left[(\lambda_1-\lambda_2)\left[3 + \frac{2}{\mathscr{R}}(1+\lambda)(\lambda_1\lambda_2-\lambda)\right] \right.$$
$$\left. + 4t[(1+\lambda)(\lambda_2-\lambda\lambda_1) - \lambda_2^2(\lambda_1+\lambda_2)]\right]$$
$$+ (1-e^{-2t(1-\lambda^2)})\left[1 + \frac{2}{\mathscr{R}}(1-\lambda)(\lambda_1\lambda_2-\lambda) + \frac{4t}{\lambda_1+\lambda_2}[(1-\lambda)(\lambda_2-\lambda\lambda_1)\right.$$
$$\left.\left. + \lambda_2^2(\lambda_1-\lambda_2)]\right]\right\}. \tag{5.4}$$

We remark here that (i) for $t=0$, $\overline{|S_2|^2}=0$, so that only $\overline{|S_1|^2}$ contributes to the GOE limit, while (ii) for $t \to \infty$, $\overline{|S_1|^2} = \overline{|S_2|^2}$; i.e., the two terms give equal contributions to the GUE limit.

The S-matrix result above can be converted into a conductance by virtue of Eq. (1.2). We define the quantity δg to represent the positive deviation of the mean conductance $\bar{g}$ from the GUE limit $t \to \infty$; i.e.,

$$\delta g(t, M) = \bar{g}(t \to \infty, M) - \bar{g}(t, M). \tag{5.5}$$

5.1. *Analytical Results for Special Cases*

For general values of t, one must resort to a numerical evaluation of Eqs. (5.1)–(5.4). However, various special cases can be treated analytically. Here, we shall quote the results, relegating details of the derivations to Appendices C and D. (a) In the GUE limit ($t \to \infty$), one integration drops out to yield

$$\overline{|S_{ab}|^2} = \int_{-1}^{+1} d\lambda \int_1^{\infty} d\lambda_1 \left(\frac{1+\lambda}{1+\lambda_1}\right)^M \frac{1}{(\lambda_1-\lambda)^2}\left[\frac{1-\lambda}{1+\lambda} + \frac{\lambda_1-1}{\lambda_1+1}\right]$$
$$= \frac{1}{M}. \tag{5.6}$$

The corresponding value of the mean conductance in this limit is then

$$\bar{g}_{\text{GUE}}(M) = M/2, \tag{5.7}$$

in agreement with Eq. (1.8). The integral in Eq. (5.6) does indeed correctly represent the GUE limit of $\overline{|S_{ab}|^2}$. This was worked out separately along the lines of Ref. [14], replacing the GOE Hamiltonian ensemble used there by a GUE Hamiltonian ensemble. (b) In the GOE limit ($t=0$), one has, for $a \neq b$,

$$\overline{|S_{ab}|^2} = \frac{1}{2}\int_{-1}^{+1} d\lambda \int_1^{\infty} d\lambda_1 \int_1^{\infty} d\lambda_2 \left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^M \frac{1-\lambda^2}{(\lambda^2+\lambda_1^2+\lambda_2^2-2\lambda\lambda_1\lambda_2-1)^2}$$
$$\times\left[\frac{1-\lambda}{1+\lambda}+\frac{\lambda_1^2+\lambda_2^2-2}{(\lambda_1+\lambda_2)^2}\right]$$
$$=\frac{1}{M+1}. \tag{5.8}$$

This leads to a mean conductance for zero field that is given by

$$\bar{g}_{\mathrm{GOE}}(M) = \frac{M^2}{2(M+1)}. \tag{5.9}$$

We have also checked that, after a suitable change of integration variables, the integral formula above agrees with Eq. (8.10) of Ref. [14] upon setting $T_c = 1$ for all c and considering $a \neq b$. Details are provided in Appendix C. (c) One can also perform a large-t expansion of Eqs. (5.1)–(5.4). For values $M \geqslant 2$, one obtains

$$\overline{|S_{ab}|^2} = \frac{1}{M} - \frac{M}{M^2-1}\frac{1}{8t} + O\left(\frac{1}{t^2}\right). \tag{5.10}$$

The corresponding result for the large-t tail of the weak-localization term is

$$\delta g(t, M) = \frac{M^3}{M^2-1}\frac{1}{16t} + O\left(\frac{1}{t^2}\right). \tag{5.11}$$

The derivation of this result, which is not entirely straightforward, is outlined in Appendix D.

5.2. *General Magnetic Field*

For general values $0 < t < \infty$, the eigenvalue integrations in Eqs. (5.2), (5.4) can no longer be carried out analytically. However, progress can made examining the implications of assuming that $\delta g(t, M)$ is indeed Lorentzian in the magnetic field: $B \sim \phi/\phi_0 \sim \sqrt{t}$. Accordingly, we write

$$\delta g(t, M) = \frac{1}{a+bt} = \frac{1/a}{1+(b/a)\,t}. \tag{5.12}$$

The unknown parameters a and b can be determined completely from the three special cases discussed above. To find a, we set $t=0$ in Eq. (5.12). This gives

$$\begin{aligned}\frac{1}{a}&=\delta g(0, M)\\&=\bar{g}(t\to\infty, M)-\bar{g}(0, M)\\&=\frac{M}{2(M+1)}\end{aligned} \tag{5.13}$$

with the help of Eqs. (5.5) and (5.6). To find b, we expand Eq. (5.12) in powers of $1/t$, which yields

$$\delta g(t, M)=\frac{1}{bt}\left[\frac{1}{1+a/(bt)}\right]=\frac{1}{bt}+\cdots. \tag{5.14}$$

Thus $1/b$ is the coefficient of $1/t$ in $\delta g(t, M)$ as given in Eq. (5.11). Consequently, we see that

$$\frac{b}{a}=\frac{8(M-1)}{M^2}. \tag{5.15}$$

Substitution of Eqs. (5.13) and (5.15) into Eq. (5.12) gives us the result for the expected Lorentzian:

$$\delta g(t, M)=\frac{M}{2(M+1)}\left[1\Big/\left(1+\frac{8(M-1)}{M^2}t\right)\right]. \tag{5.16}$$

Although the conjecture (5.16) will have to be verified numerically, there does exist a particular limit in which the correctness of the Lorentzian form can be established analytically. This comes from observing that for large M, the weak localization term, as given by Eq. (5.16), depends on M and t only through the ratio t/M. It is possible to expand the exact integral representation of $\delta g(t, M)$ in the simultaneous limits $t\to\infty$ and $M\to\infty$ such that the ratio t/M is kept fixed and arbitrary. As is demonstrated in Appendix E, the resulting eigenvalue integrations can be performed analytically. In leading order, one reproduces the GUE limit of Eq. (5.7). The weak-localization term is then given as

$$\delta g(t, M)=f_0(t/M)+M^{-1}f_1(t/M)+M^{-2}f_2(t/M)+\cdots. \tag{5.17}$$

The leading correction reads

$$f_0=\frac{M}{2(M+8t)}. \tag{5.18}$$

This result is consistent with Eq. (5.16) and confirms that, for large values of M, the leading correction to the GUE limit is a Lorentzian in the magnetic field $B \sim \sqrt{t}$. The next two corrections read

$$\frac{f_1}{M} = -\frac{M}{2(M+8t)^2},$$
$$\frac{f_2}{M^2} = \frac{(8t)^4 + 4(8t)^3 M + 13(8t)^2 M^2 + 16tM^3 + M^4}{2M(M+8t)^5}. \quad (5.19)$$

One can easily check that these results are in agreement with the expressions for the GUE limit ($t/M \to \infty$) and the GOE limit ($t/M \to 0$). They also agree with the large-t tail given in Eq. (5.11).

6. Numerical Analysis

In order to verify whether Eq. (5.16) is indeed a good representation of Eqs. (5.1)–(5.4) over the whole range of values of t and M, we should compute

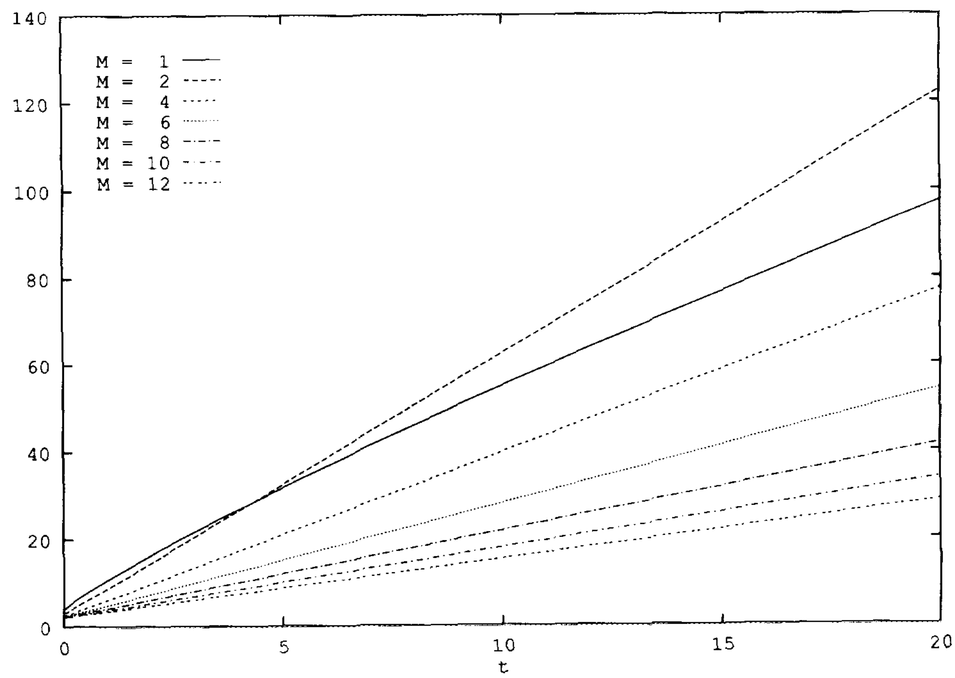

FIG. 5. Graph of the reciprocal of the weak-localization term δg^{-1} versus the magnetic-field parameter t for $M = 1, 2, 4, 6, 8, 10, 12$ on a scale $(0 \leqslant t \leqslant 20)$.

$\overline{|S_{ab}|^2}$ numerically as a function of these parameters. This data is most easily digested if we plot the curves of reciprocal weak-localization versus t, i.e.,

$$\delta g^{-1} = \frac{2}{M^2}\left[\frac{1}{M} - \overline{|S_{ab}|^2}\right]^{-1} \quad \text{vs} \quad t, \tag{6.1}$$

for various values of M. According to the Lorentizian hypothesis, we expect to find

$$\delta g^{-1}(t, M) = a(M) + b(M)\,t, \tag{6.2}$$

i.e., a straight line in t for each value of M.

Unfortunately, the exact integral expression for $\overline{|S_{ab}|^2}$ as given by Eqs. (5.1)–(5.4) is not directly amenable to numerical evaluation. It must first be subjected to a number of changes of integration variable in order to improve the behaviour of the integrands. This procedure is outlined in Appendix F.

Some numerical results for the curves specified by Eq. (6.1) are presented in Figs. 5 and 6. One can make the following observations: (i) For $M = 2, 3, 4, 5, \ldots$ and $t \lesssim 100$, the numerical plots agree with the "theoretical" straight lines to better than 1 part in 10^2, and the degree of agreement improves as M is increased, consistent

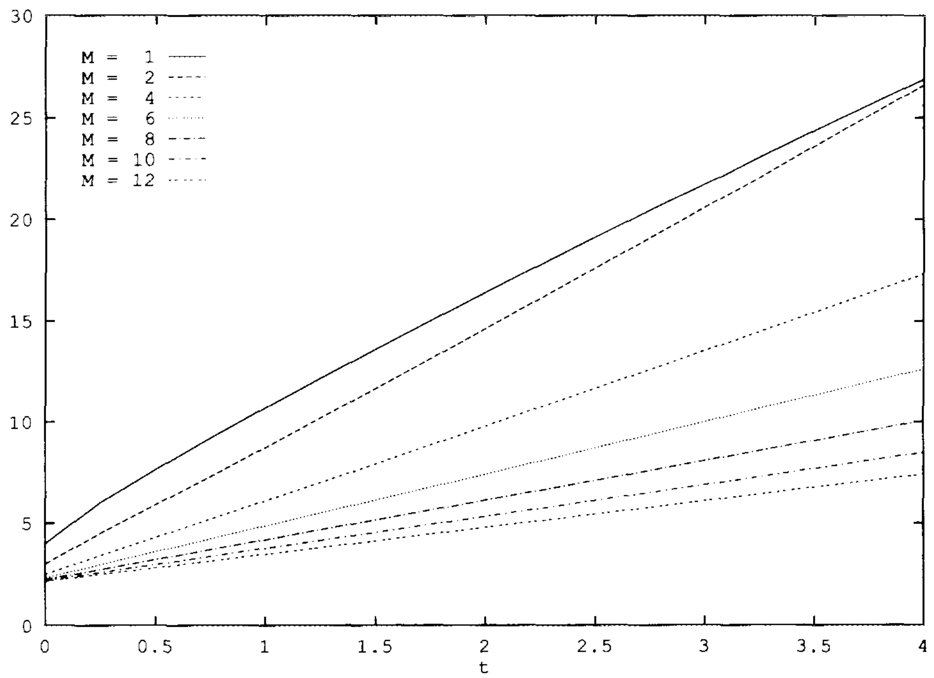

FIG. 6. Graph of the reciprocal of the weak-localization term δg^{-1} versus the magnetic-field parameter t for $M = 1, 2, 4, 6, 8, 10, 12$ on a smaller scale $(0 \leqslant t \leqslant 4)$.

with Eq. (5.18). (ii) Odd values of M are unphysical, although the threefold integral remains well-defined. This holds true even for $M = 1$. We observe, however, that the $M = 1$ curve deviates significantly from a straight line. Moreover, Eq. (5.16) does not even make a sensible prediction for what straight line could have been expected in this case. To illustrate the extent of the deviation, we plot in Fig. 7 the weak-localization term δg as a function of $\sqrt{t}$ for $M = 1$ and compare it with the corresponding Lorentzians obtained from fitting to the large-$\sqrt{t}$ tail of the $M = 1$ curve, and also its FWHM. The inference that one draws from all this is that, viewing $\delta g(t, M)$ as a continuous function of the real variable M, a transition to almost Lorentzian behaviour happens somewhere in the region $1 < M < 2$.

One technical point, especially relevant to the case $M = 1$, that is worth mentioning is the following: For given M, the quantity $\overline{|S_1|^2}$ varies monotonically from the GOE value of $1/(M+1)$ at $t = 0$ to half the GUE value, viz. $1/(2M)$, in the limit $t \to \infty$. Thus for $M = 1$, it should remain constant with a value of $\frac{1}{2}$ despite its complicated structure in the variable t. This is precisely what is observed numerically.

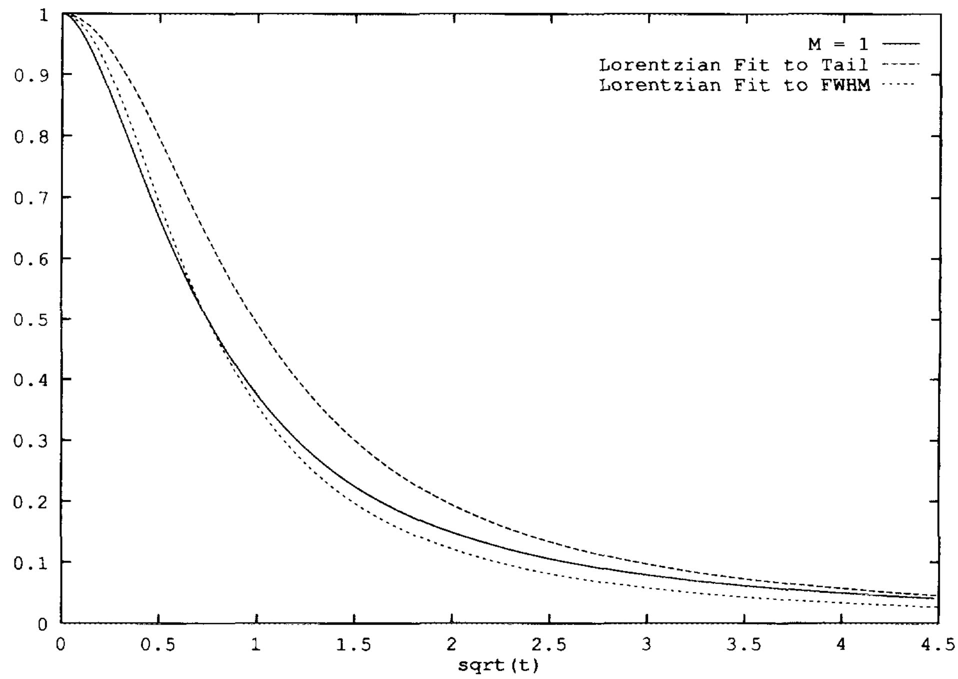

FIG. 7. Plot of the weak-localization term δg as a function of $\sqrt{t}$ for the case $M = 1$, compared with the corresponding Lorentzians obtained from fitting to the large-$\sqrt{t}$ tail of the $M = 1$ curve and to its FWHM.

7. Discussion

To determine the magnetic-field scale on which the GOE → GUE transition occurs, let us look at the full width at half maximum (FWHM), which we shall denote by $F_{1/2}(M)$, of the δg versus $\sqrt{t}$ curve: $\delta g = 1/[a + b(\sqrt{t})^2]$. From Eq. (5.16), it is easy to see that

$$F_{1/2}(M) = 2\sqrt{a/b} = \frac{M}{\sqrt{2(M-1)}} \underset{M \gg 1}{\sim} \sqrt{M/2}. \tag{7.1}$$

Thus we conclude that the GOE → GUE transition is delayed as the number of open channels increases. We should also observe from Eq. (5.16) that the limits $t \to \infty$ and $M \to \infty$ do not commute:

$$\begin{aligned} &\lim_{t \to \infty} \delta g(t, M) = 0 \qquad \text{for all} \quad M, \\ &\lim_{M \to \infty} \delta g(t, M) = \tfrac{1}{2} \qquad \text{for all} \quad t. \end{aligned} \tag{7.2}$$

In the former case, one trivially remains in the GUE limit whatever the value of M, while in the latter, no crossover is observed, with the GOE weak-localization term holding for all values of t.

This phenomenon can be related to two extreme cases in problems involving quasi-one-dimensional rings. When one considers an isolated quasi-one-dimensional ring (hence not coupled to external channels) such as in the calculation of persistent currents [5], the transition from GOE to GUE sets in very rapidly with increasing flux angle—at a value around $\phi/\phi_0 \sim [d/(\pi E_c)]^{1/2}$, where E_c denotes the Thouless energy. On the other hand, in the problem of AAS oscillations [26] in the same quasi-one-dimensional ring now coupled to two leads carrying a large number of channels M, one computes the weak-localization term as an expansion in terms of diffusions and Cooperons [3]. Here, one does not observe a transition to GUE behaviour for any value of the flux angle. The oscillations of the magneto-conductance persist for all values of ϕ/ϕ_0, as the Cooperon (a periodic function in ϕ/ϕ_0) is never suppressed. The hypothesis that the GOE → GUE transtion is delayed with increasing channel number clearly renders these two distinct observations consistent.

The broadening of the Lorentzian with increasing channel number implied by Eq. (5.16) can be simply understood in terms of the two relevant time scales that enter the problem. (i) We have τ_{mix}—the average time taken by the time-reversal violating perturbation (due to B) to mix the GOE levels. The width associated with this time scale is

$$\Gamma_{\text{mix}} = \frac{2\pi}{d}\left[\frac{t}{N}\overline{|H_{\mu\nu}^{(A)}|^2}\right] = \frac{2\pi t}{d}\cdot\frac{\lambda^2}{N^2}, \tag{7.3}$$

in which case

$$\tau_{\text{mix}} = \frac{h}{\Gamma_{\text{mix}}} = \frac{\pi h}{2td}, \tag{7.4}$$

since $d = \pi\lambda/N$. Thus, a stronger magnetic field requires a shorter time to mix the levels. (ii) We also have τ_{dec}—the decay time of states in the internal region due to the coupling with the leads. The associated width is given by

$$\Gamma_{\text{dec}} = \frac{Md}{2\pi}, \tag{7.5}$$

and this leads to

$$\tau_{\text{dec}} = \frac{h}{\Gamma_{\text{dec}}} = \frac{2\pi h}{Md}, \tag{7.6}$$

from which we see that for more open channels, the electrons spend less time within the stadium. Therefore, an increase in the number of open channels implies a decrease in τ_{dec}, which means that the same degree of time-reversal violation will require a smaller τ_{mix}; and this is achieved by a stronger magnetic field. The ratio $\Gamma_{\text{mix}}/\Gamma_{\text{dec}} \sim t/M$ provides a measure of the time-reversal violation, and we note that for $M \gg 1$, δg depends on t only via the combination t/M. This feature is not explicit in the semiclassical result of Ref. [12] but can be inferred indirectly [27, 28].

If one considers the problem of calculating $\delta g(t, M)$ in the framework of the asymptotic expansion of Refs. [1, 3], which essentially proceeds in inverse powers of M, then one is able to identify analogues of the diffusion and Cooperon as $\Pi_{\text{D}}^{-1} = M$ and $\Pi_{\text{C}}^{-1} = M + 8t$, respectively. Equation (5.16) has a very simple interpretation in terms of these quantities: If one considers M to be large (compared with unity) but t/M not necessarily small, then one can write

$$\delta g(t, M) \underset{M \gg 1}{\sim} 2\left(\frac{M}{2}\right)\left(\frac{M}{2}\right)\Pi_{\text{D}}\Pi_{\text{C}}, \tag{7.7}$$

where the first three factors represent the spin degeneracy and the summations over the incoming and outgoing channels. In this approach, a diffusion factor is always implied by the structure of the lowest-order contribution to the source term and reflects the coupling with the leads. With increasing t, the Cooperon contribution is damped out and, with it, δg. One should also note that the RHS of Eq. (7.7) corresponds to the first term of the expansion given in Eq. (5.17).

We have treated the stadium billiard strictly as a one-particle system in which classical chaos and magnetic flux combine to cause the GOE → GUE crossover transition. In actual fact, we should have treated the stadium as a problem involving a large number (say A) of non-interacting electrons. In such a system, the eigenvalues of the many-body Hamiltonian $H = \sum_{j=1}^{A} H(j)$ will *not* obey GOE (or GUE) statistics, even if the eigenvalues of each of the single-particle Hamiltonians

$H(j), j = 1, ..., A$, do. Does this fact necessitate a rethinking of our method and of our results? The answer is no. Indeed, using the argument of Ref. [29], one can easily show that for non-interacting fermions, it is legitimate to calculate the single-particle scattering matrix without paying attention to the presence of the other fermions.

It is of interest to compare the suppression of weak localization by a magnetic field B in the ballistic and in the diffusive regimes. In the ballistic regime, we have found that for $M \gg 1$, the crossover behaviour is governed by the dimensionless parameter $M^{-1}(\phi/\phi_0)^2$. In extended diffusive samples, the analogous parameter is $(BD\tau_\varphi/\phi_0)^2$ [30]. Here, D is the diffusion constant, and τ_φ is the inelastic mean free time. The factor $BD\tau_\varphi$ can be interpreted as the flux through the area $D\tau_\varphi$. This is the area covered by diffusive motion during the time interval τ_φ. In the ballistic case, $D\tau_\varphi$ is replaced by the actual area A of the stadium, and the crossover is delayed by the additional factor M^{-1}. This factor accounts for the presence of open channels and, hence, for the actual width of the levels in the stadium. In the diffusive regime, the corresponding width is given by $\hbar/\tau_\varphi$ and is already contained in the factor $D\tau_\varphi$. In finite quasi-one-dimensional diffusive systems with dimensions smaller than the phase-coherence length L_φ, the magnitude of the weak-localization correction ($\frac{2}{3}$) is larger than our maximum value ($\frac{1}{2}$) for δg. The area is given by the area of the sample.

Finally, let us attempt a comparison of our theoretical result (5.16) with experimental data. There are several reasons why a direct comparison of our Eq. (5.16) with the results of Ref. [7] for $\bar{g}$ is not possible. First, the stadium used in the experiment possesses a twofold reflection symmetry. Hence, in the closed stadium there exist classes of states pertaining to different quantum numbers with respect to this symmetry. This fact is not taken into account in our modelling of the Hamiltonian. Second, the stadium is known [31] to have a family of "bouncing ball" orbits which might obscure the connection with our random-matrix model. In view of the way the leads are attached to the stadium, it is likely that these orbits affect the experiment only slightly. Third, the data of Ref. [7] were obtained by averaging over the gate voltage or, equivalently, over the number of channels and the area of the stadium. Finally, the data show a marked dependence on temperature which we do not take into account. For purposes of orientation, we nonetheless take the data of Ref. [7] at face value. For large B, the value of $\bar{g}$ is close to 3. This suggests $M \approx 6$, recalling that $\bar{g}_{\mathrm{GUE}}(M) = M/2$. Insertion of this value of M into Eq. (5.9) for the zero-field value of δg yields a theoretical prediction of 0.43, to be compared with a value of $\frac{1}{3}$ found experimentally. Also, one can write

$$\sqrt{t} = k\phi/\phi_0 = (0.24)\, kab, \tag{7.8}$$

where a is the area of the stadium measured in $(\mu\mathrm{m})^2$ and b is the magnetic field in units of millitesla. With the aid of Eq. (7.1), this gives the theoretical FWHM in b to be

$$\Delta b_{\mathrm{FWHM}} = \frac{1}{0.24ka} \sqrt{M^2/2(M-1)} \simeq 2.7, \tag{7.9}$$

for $M=6$, $a \simeq 0.5$ $(\mu\text{m})^2$, and taking $k \approx 5.8$. This value of k results from esitmating the number n of electrons in the stadium to be ~ 2000 and using the scaling relation

$$k_n = \left(\frac{n}{400}\right)^{1/4} k_{400} = 3.87 \left(\frac{n}{400}\right)^{1/4}, \tag{7.10}$$

as discussed in Section 3. Both the values for the height of the Lorentzian and the FWHM differ somewhat from what we read off the experimentally fitted Lorentzian shown as a dotted line in Fig. 3b of Ref. [7], namely, $\frac{1}{3}$ and 5 mT, respectively. The discrepancy is consistent with the presence in the experimental data of effects, such as finite temperature and inelastic scattering, that will tend to wash out coherent phenomena, and that manifest themselves as a flattening of the peak and a broadening of the width.

8. Conclusions

We have derived a universal expression for the flux and channel-number dependence of weak localization, δg, in classically chaotic ballistic microstructures. For $M \geqslant 2$, $\delta g(t, M)$ is almost Lorentzian in $\sqrt{t} \sim B$. The FWHM of $\delta g(t, M)$ versus $\sqrt{t}$ grows with channel number essentially as $\sqrt{M/2}$, so that the GOE $\rightarrow$ GUE crossover occurs at $t \sim M/8$. This fact is arguably the most interesting result of this paper: An increase in the number of channels causes an increasing delay in the GOE $\rightarrow$ GUE crossover transition.

It is remarkable that for values of M as small as two, $\delta g(t, M)$ is almost Lorentzian in $\sqrt{t} \sim B$. This is reminiscent of results for the S-matrix auto-correlation function versus energy in the GOE limit. In Ref. [10], it was found that this function is close to Lorentzian even for small M, provided all channels have $T_c = 1$. For "inequivalent" channels, i.e., channels with different T_c's, marked deviations from the Lorentzian form occur for M as large as 10 [10]. We speculate that, similarly, gating one of the leads connected to the stadium may significantly affect the functional dependence of $\delta g(t, M)$ on t.

Appendix A

In this appendix, we collect the various constant matrices that appear throughout the paper. Matrices arising from the graded structure of the theory and the symmetry-breaking due to the presence of advanced and retarded components are given by

$$k^{\alpha\alpha'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\alpha\alpha'}, \qquad L_{pp'} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}_{pp'}, \qquad s_{pp'} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -k \end{pmatrix}_{pp'}. \tag{A.1}$$

Matrices associated with the doubling of dimensionality in the GOE problem relative to the GUE are given by

$$\Lambda_{rr'} = \begin{pmatrix} 0 & 1 \\ k & 0 \end{pmatrix}_{rr'}, \qquad C = s \otimes \Lambda. \tag{A.2}$$

Matrices arising from the presence of magnetic field and the source terms are given by

$$\tau^3_{rr'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{rr'}, \qquad I_{pp'}(1) = \begin{pmatrix} -k & 0 \\ 0 & 0 \end{pmatrix}_{pp'}, \qquad I_{pp'}(2) = \begin{pmatrix} 0 & 0 \\ 0 & -k \end{pmatrix}_{pp'}. \tag{A.3}$$

In all cases, the explicit indices indicate the space to which the displayed block structure pertains.

Appendix B

In Efetov's parametrization, (used throughout this appendix), one writes $Q = UD\bar{U}$, where

$$D = \begin{pmatrix} \cos\hat{\theta} & i\sin\hat{\theta} \\ -i\sin\hat{\theta} & -\cos\hat{\theta} \end{pmatrix}, \qquad \hat{\theta} = \begin{pmatrix} \hat{\theta}_B & 0 \\ 0 & \hat{\theta}_F \end{pmatrix}, \tag{B.1}$$

with

$$\hat{\theta}_B = i \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_2 & \theta_1 \end{pmatrix}, \qquad \hat{\theta}_F = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}, \qquad U = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \qquad \bar{U} = \begin{pmatrix} \bar{u} & 0 \\ 0 & \bar{v} \end{pmatrix}, \tag{B.2}$$

and we have the relations $\bar{u} = u^+$, $\bar{v} = kv^+k$. The rows and columns of matrices of dimensions 8, 4, and 2 are labelled by the indices (p, α, r), (α, r), and r, respectively: the indices follow in lexicographical order. (With $\alpha = 0, 1$, this order differs from that of Efetov [15].) The 4×4 matrix u is itself expressed as a product of two matrices $u = u_1 u_2$ (whence $\bar{u} = \bar{u}_2 \bar{u}_1$), where

$$u_1 = \begin{pmatrix} 1 - 2\bar{\eta}\eta + 6(\bar{\eta}\eta)^2 & -2(1 - 2\bar{\eta}\eta)\,\bar{\eta} \\ 2\eta(1 - 2\bar{\eta}\eta) & 1 - 2\eta\bar{\eta} + 6(\eta\bar{\eta})^2 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}. \tag{B.3}$$

Here,

$$\eta = \begin{pmatrix} \eta_1 & \eta_2 \\ -\eta_2^* & -\eta_1^* \end{pmatrix}, \qquad \bar{\eta} = \eta^+ = \begin{pmatrix} \eta_1^* & \eta_2 \\ \eta_2^* & \eta_1 \end{pmatrix} \tag{B.4}$$

and

$$F_1 = \exp(i\phi\tau^3), \qquad F_2 = \begin{pmatrix} w & z \\ -z^* & w^* \end{pmatrix}, \tag{B.5}$$

where $|w|^2+|z|^2=1$. We note that F_2 is an SU(2) matrix, which can be expressed in terms of unconstrained parameters according to

$$w=\frac{(1-im)^2-|m_1|^2}{1+m^2+|m_1|^2}, \qquad z=\frac{-2im_1^*}{1+m^2+|m_1|^2}. \tag{B.6}$$

Similarly, we write $v=v_1v_2$ (whence $\bar{v}=\bar{v}_2\bar{v}_1$), where

$$v_1=\begin{pmatrix} 1+2\bar{\rho}\rho+6(\bar{\rho}\rho)^2 & -2i(1+2\bar{\rho}\rho)\bar{\rho} \\ 2i\rho(1+2\bar{\rho}\rho) & 1+2\rho\bar{\rho}+6(\rho\bar{\rho})^2 \end{pmatrix}, \qquad v_2=\begin{pmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{pmatrix}. \tag{B.7}$$

Here

$$\rho=\begin{pmatrix} \rho_1 & \rho_2 \\ -\rho_2^* & -\rho_1^* \end{pmatrix}, \qquad \bar{\rho}=\rho^\dagger=\begin{pmatrix} \rho_1^* & \rho_2 \\ \rho_2^* & \rho_1 \end{pmatrix}, \tag{B.8}$$

and $\Phi_1=\exp(i\chi\tau^3)$, $\Phi_2=\mathbf{1}$. The normalization of u's and v's is given by $\bar{u}u=\bar{v}v=1$, $\bar{u}_ju_j=\bar{v}_jv_j=1$, for $j=1, 2$.

In the parametrization given above, $\theta_1\theta_2$, θ, m, ϕ, χ are real commuting variables, m_1 is a complex commuting variable, while η_1, η_1^*, η_2, η_2^*, ρ_1, ρ_1^*, ρ_2, ρ_2^* are all anticommuting variables. If we also introduce $\lambda_1=\cosh\theta_1$, $\lambda_2=\cosh\theta_2$, and $\lambda=\cos\theta$, then in terms of these final variables, the integration measure on the space of Q-matrices is given by

$$\mathscr{D}Q=\frac{2(1-\lambda^2)}{(64\pi\mathscr{R})^2}\,d\lambda_1\,d\lambda_2\,d\lambda\,\mathscr{D}F_2\,d\phi\,d\chi\,d\eta_1\,d\eta_1^*\,d\eta_2\,d\eta_2^*\,d\rho_1\,d\rho_1^*\,d\rho_2\,d\rho_2^*, \tag{B.9}$$

where $\mathscr{R}=\lambda_1^2+\lambda_2^2+\lambda^2-2\lambda\lambda_1\lambda_2-1$ and $\mathscr{D}F_2$ denotes the SU(2) integration measure

$$\mathscr{D}F_2=\frac{4}{\pi^2(1+m^2+|m_1|^2)^3}\,dm\,d(\operatorname{Re} m_1)\,d(\operatorname{Im} m_1). \tag{B.10}$$

One should note that Eq. (B.9) assumes Efetov's convention [15] for integrals over the anticommuting variables, $\int d\chi\,\chi=1$, rather than the convention of Ref. [14], namely $\int d\chi\,\chi=(2\pi)^{-1/2}$. The integration region corresponds to $1<\lambda_1, \lambda_2<\infty$, $-1<\lambda<1$, $-\infty<m$, $\operatorname{Re} m_1$, $\operatorname{Im} m_1<\infty$, $0<\phi$, $\chi<\pi$.

To perform the Grassmann integration, we recall that only that part of the integrand which is of the highest (eighth) order in the anticommuting variables contributes [5]. The M-dependent term of the integrand depends solely on the generalized eigenvalues θ_1, θ_2, θ. By diagonalizing the matrix $\hat{\theta}$, we find that

$$\exp\left\{-\frac{1}{2}M\operatorname{trg}\ln(1+QL)\right\}=\operatorname{detg}^{-M/2}(1+QL)=\left(\frac{1+\cos\theta}{\cosh\theta_1+\cosh\theta_2}\right)^M. \tag{B.11}$$

After making the substitution

$$(1+QL)^{-1}=\frac{1}{2}U\begin{pmatrix}1 & i\sin\hat{\theta}(1+\cos\hat{\theta})^{-1}\\ i\sin\hat{\theta}(1+\cos\hat{\theta})^{-1} & 1\end{pmatrix}\bar{U}, \tag{B.12}$$

we find, for the symmetry-breaking and source terms, the explicit forms

$$\begin{aligned}\operatorname{trg}(Q\tau^3Q\tau^3)=&\operatorname{trg}[(\cos\hat{\theta}\bar{u}_2\cdot\bar{u}_1\tau^3u_1\cdot u_2)^2+2\sin\hat{\theta}\bar{u}_2\cdot\bar{u}_1\tau^3u_1\cdot u_2\sin\hat{\theta}\bar{v}_2\cdot\bar{v}_1\tau^3v_1\cdot v_2\\&+(\cos\hat{\theta}\bar{v}_2\cdot\bar{v}_1\tau^3v_1\cdot v_2)^2],\end{aligned} \tag{B.13}$$

$$\begin{aligned}&\operatorname{trg}[(1+QL)^{-1}I_1(1)(1+QL)^{-1}I_1(2)]\\&\quad=-\tfrac{1}{4}\operatorname{trg}[(1+\cos\hat{\theta})^{-1}\sin\hat{\theta}\bar{u}_2\cdot\bar{u}_1ku_1\cdot u_2(1+\cos\hat{\theta})^{-1}\sin\hat{\theta}\bar{v}_2\cdot\bar{v}_1kv_1\cdot v_2],\end{aligned} \tag{B.14}$$

$$\begin{aligned}&\operatorname{trg}[(1+QL)^{-1}I_2(1)(1+QL)^{-1}I_2(2)]\\&\quad=-\tfrac{1}{4}\operatorname{trg}[(1+\cos\hat{\theta})^{-1}\sin\hat{\theta}\bar{u}_2\cdot\bar{u}_1k\tau^3u_1\cdot u_2(1+\cos\hat{\theta})^{-1}\sin\hat{\theta}\bar{v}_2\cdot\bar{v}_1k\tau_3v_1\cdot v_2].\end{aligned} \tag{B.15}$$

The anticommuting variables enter the traces in Eqs. (B.13)–(B.15) only via the matrices $\bar{u}_1\mu u_1$ and $\bar{v}_1\mu v_1$, with $\mu=\tau^3$, k, $k\tau^3$. By employing Eqs. (B.3)–(B.8), we find that these matrices have the form

$$\bar{u}_1\mu u_1=U_f\mu_fU_f^\dagger,\qquad \bar{v}_1\mu v_1=U_g\mu_gU_g^\dagger, \tag{B.16}$$

where U_f, U_g are the unitary matrices ($U_fU_f^\dagger=U_gU_g^\dagger=1$) defined by

$$U_f=\begin{pmatrix}1&0\\0&\exp(\gamma_f)\end{pmatrix},\qquad U_g=\begin{pmatrix}1&0\\0&\exp(\gamma_g)\end{pmatrix}, \tag{B.17}$$

with

$$\gamma_f=4\begin{pmatrix}0&-\eta_1\eta_2\\\eta_1^*\eta_2^*&0\end{pmatrix},\qquad \gamma_g=4\begin{pmatrix}0&\rho_1\rho_2\\-\rho_1^*\rho_2^*&0\end{pmatrix}. \tag{B.18}$$

We note that (i) the matrix U_g commutes with $\hat{\theta}$, v_2, and $\bar{v}_2$; (ii) the matrix τ_f^3 depends solely on η_2, η_2^*; and (iii) the matrix τ_g^3 depends solely on ρ_2, ρ_2^*. Making use of Eq. (B.16) and (i), we rewrite the traces standing on the RHS of Eqs. (B.13)–(B.15) as follows:

$$\begin{aligned}\operatorname{trg}(Q\tau^3Q\tau^3)=&\operatorname{trg}[(\cos\hat{\theta}U_g^\dagger\bar{u}_2U_f\tau_f^3U_f^\dagger u_2U_g)^2\\&+2\sin\hat{\theta}U_g^\dagger\bar{u}_2U_f\tau_f^3U_f^\dagger u_2U_g\sin\hat{\theta}\bar{v}_2\tau_g^3v_2+(\cos\hat{\theta}\bar{v}_2\tau_g^3v_2)^2],\end{aligned} \tag{B.19}$$

$$\begin{aligned}&\operatorname{trg}[(1+QL)^{-1}I_1(1)(1+QL)^{-1}I_1(2)]\\&\quad=-\tfrac{1}{4}\operatorname{trg}[(1+\cos\hat{\theta})^{-1}\sin\hat{\theta}U_g^\dagger\bar{u}_2U_fk_fU_f^\dagger u_2U_g(1+\cos\hat{\theta})^{-1}\sin\hat{\theta}\bar{v}_2k_gv_2],\end{aligned} \tag{B.20}$$

$$\operatorname{trg}[(1+QL)^{-1} I_2(1)(1+QL)^{-1} I_2(2)]$$
$$= -\tfrac{1}{4}\operatorname{trg}[(1+\cos\hat\theta)^{-1}\sin\hat\theta U_g^\dagger \bar u_2 U_f(k\tau_3)_f U_f^\dagger u_2 U_g (1+\cos\hat\theta)^{-1}$$
$$\times \sin\hat\theta \bar v_2 (k\tau_3)_g v_2]. \qquad (B.21)$$

The calculation can be greatly simplified by using a trick introduced in Section 4.1 of Ref. [5], which involves passing from the SU(2) commuting variables specifying u_2 to new (primed) SU(2) commuting variables specifying u_2' defined by

$$u_2' = U_f^\dagger u_2 U_g, \qquad \bar u_2' = U_g^\dagger \bar u_2 U_f, \qquad (B.22)$$

with u_2' being the same function of the variables ϕ, m', m_1' as u_2 is of the variables ϕ, m, m_1. The new commuting variables m, m_1' are functions of the old commuting variables m, m_1 and of the products $\eta_1\eta_2$, $\eta_1^*\eta_2^*$, $\rho_1\rho_2$, and $\rho_1^*\rho_2^*$. The Berezinian of this transformation of variables is equal to unity. For simplicity, the primes will be suppressed in the sequel. We note also that the traces depend on the anticommuting variables only via the matrices μ_f and μ_g. Furthermore, the exponential symmetry breaking term depends only on the anticommuting variables η_2, η_2^*, ρ_2, ρ_2^*:

$$\begin{aligned}
\operatorname{trg}(Q\tau^3 Q\tau^3) &= \operatorname{trg}[(\cos\hat\theta \bar u_2 \tau_f^3 u_2)^2 + 2\sin\hat\theta \bar u_2 \tau_f^3 u_2 \sin\hat\theta \bar v_2 \tau_g^3 v_2 \\
&\quad + (\cos\hat\theta \bar v_2 \tau_g^3 v_2)^2] \\
&= 8(\sinh^2\theta_2 + |z|^2\sin^2\theta) + 64(\eta_2\eta_2^* - \rho_2\rho_2^*) \\
&\quad \times (\cosh^2\theta_2 - \cosh\theta_1\cosh\theta_2\cos\theta - |z|^2\sin^2\theta) \\
&\quad - 64[\eta_2\rho_2\cosh\theta_1\sinh\theta_2\sin\theta z^*\exp(-i(\phi+\chi)) + \text{c.c.}] \\
&\quad + 64[\eta_2\rho_2^*\sinh\theta_1\cosh\theta_2\sin\theta w^*\exp(-i(\phi-\chi)) + \text{c.c.}] \\
&\quad + 256\eta_2\eta_2^*\rho_2\rho_2^*[\cosh^2\Theta_1 - \cosh^2\theta_2 + (|w|^2-|z|^2)\sin^2\theta]. \qquad (B.23)
\end{aligned}$$

Thus,

$$\begin{aligned}
&\exp\{-\tfrac{1}{4}t\operatorname{trg}(Q\tau^3 Q\tau^3)\} \\
&\quad = \exp\{-2t(\sinh^2\theta_2 + |z|^2\sin^2\theta)\} \\
&\qquad \times \{1 - 16t(\eta_2\eta_2^* - \rho_2\rho_2^*)(\cosh^2\theta_2 - \cosh\theta_1\cosh\theta_2\cos\theta - |z|^2\sin^2\theta) \\
&\qquad + 16t[\eta_2\rho_2\cosh\theta_1\sinh\theta_2\sin\theta\, z^*\exp(-i(\phi+\chi)) + \text{c.c.}] \\
&\qquad - 16t[\eta_2\rho_2^*\sinh\theta_1\cosh\theta_2\sin\theta\, w^*\exp(-i(\phi-\chi)) + \text{c.c.}] \\
&\qquad - 64t\eta_2\eta_2^*\rho_2\rho_2^*[\cosh^2\theta_1 - \cosh^2\theta_2 + (|w|^2-|z|^2)\sin^2\theta \\
&\qquad + 4t(\cosh^2\theta_2 - \cosh\theta_1\cosh\theta_2\cos\theta - |z|^2\sin^2\theta)^2 \\
&\qquad + 4t\sin^2\theta(|w|^2\sinh^2\theta_1\cosh^2\theta_2 + |z|^2\cosh^2\theta_1\sinh^2\theta_2)]\}. \qquad (B.24)
\end{aligned}$$

For working out the integral, only those parts of the source terms which, on multiplying Eq. (B.24), yield terms of the eighth order in the anticommuting variables are of interest. The explicit forms of these parts are as follows (cp = contributing part):

$$\begin{aligned}
&\operatorname{trg}[(1+QL)^{-1} I_1(1)(1+QL)^{-1} I_1(2)] \\
&\quad = -\tfrac{1}{4} \operatorname{trg}[(1+\cos\hat{\theta})^{-1} \sin\hat{\theta}\bar{u}_2 k_f u_2 (1+\cos\hat{\theta})^{-1} \sin\hat{\theta}\bar{v}_2 k_g v_2] \\
&\quad \xrightarrow{\text{cp}} -32\eta_1\eta_1^*\rho_1\rho_1^*(1-8\eta_2\eta_2^*)(1+8\rho_2\rho_2^*) \\
&\qquad \times [(\cosh^2\theta_1 + \cosh^2\theta_2 - 2)(\cosh\theta_1 + \cosh\theta_2)^{-2} \\
&\qquad + \sin^2\theta(1+\cos\theta)^{-2}],
\end{aligned} \tag{B.25}$$

$$\begin{aligned}
&\operatorname{trg}[(1+QL)^{-1} I_2(1)(1+QL)^{-1} I_2(2)] \\
&\quad = -\tfrac{1}{4} \operatorname{trg}[(1+\cos\hat{\theta})^{-1} \sin\hat{\theta}\bar{u}_2 (k\tau^3)_f u_2 (1+\cos\hat{\theta})^{-1} \\
&\qquad \times \sin\hat{\theta}\bar{v}_2 (k\tau^3)_g v_2] \\
&\quad \xrightarrow{\text{cp}} -32\eta_1\eta_1^*\rho_1\rho_1^*(\cosh\theta_1 + \cosh\theta_2)^{-1}(1+\cos\theta)^{-1} \\
&\qquad \times \{(1-16\eta_2\eta_2^*)(1+16\rho_2\rho_2^*)(\cosh\theta_1+\cosh\theta_2) \\
&\qquad \times (1-\cos\theta)(|w|^2 - |z|^2) \\
&\qquad + (\cosh\theta_1 - \cosh\theta_2)(1+\cos\theta) \\
&\qquad + [16\eta_2\rho_2 \sinh\theta_2 \sin\theta z^* \exp(-i(\phi+\chi)) + \text{c.c.}] \\
&\qquad - [16\eta_2\rho_2^* \sinh\theta_1 \sin\theta w^* \exp(-i(\phi-\chi)) + \text{c.c.}]\}.
\end{aligned} \tag{B.26}$$

Using Eqs. (B.24)–(B.26), the desired leading part of the integrand (i.e., eighth order in the anticommuting variables) can be found. This part is independent of ϕ and χ, and its dependence on the SU(2) variables is carried by the factors $x^n \exp(ax)$, where $a = t\sin^2\theta$ and $x = |w|^2 - |z|^2$. (The index n ranges over $n = 0, 1, 2$ for the source term of Eq. (B.25), and over $n = 0, 1, 2, 3$ for the source term of Eq. (B.26).) The integrals over the anticommuting variables and over the angles ϕ and χ can now be easily done. In particular, the integration over the SU(2) parameters can be carried out with the help of the formula (Section 4.2 of Ref. [5])

$$\int \mathscr{D}F_2 x^n e^{ax} = \frac{d^n}{da^n}\left(\frac{\sinh a}{a}\right) \tag{B.27}$$

for $n = 0, 1, 2, 3$. Having performed these integrations, we finally obtain $\overline{|S_{ab}|^2}$ in the form of Eqs. (5.1)–(5.4).

Appendix C

GUE Limit

We begin our discussion of the GUE limit ($t \to \infty$) by outlining the derivation of the integral appearing in Eq. (5.6). For $\overline{|S_1|^2}$, the contribution that survives the large-t limit in Eq. (5.2) comprises the term whose coefficient is $4t$, but upon neglecting the exponentials $e^{-2t(1-\lambda^2)}$, i.e.,

$$\overline{|S_1|^2} \underset{t\to\infty}{\sim} \frac{1}{2}\int_{-1}^{+1} d\lambda \int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2 \left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^M e^{-2t(\lambda_2^2-1)} \left[\frac{1-\lambda}{1+\lambda} + \frac{\lambda_1^2+\lambda_2^2-2}{(\lambda_1+\lambda_2)^2}\right] \frac{4t\lambda_2^2}{\mathscr{R}}. \tag{C.1}$$

For $t \gg 1$, this integral is dominated by the region $\lambda_2 \sim 1$. Thus we write $\lambda_2 = 1 + z$, so that

$$e^{-2t(\lambda_2^2-1)} = e^{-4tz}(1 - 2tz^2 + \cdots). \tag{C.2}$$

The rest of the integrand should also be expanded in powers of z, retaining only the lowest order contributions. This is equivalent to the remplacement $\lambda_2 \to 1$ everywhere, except in the exponential on the RHS of Eq. (C.2). We note that

$$\mathscr{R} \underset{\lambda_2\to 1}{\sim} (\lambda_1 - \lambda)^2. \tag{C.3}$$

The exponential integral over z can now be trivialy carried out, and we obtain exactly half the integral expression of Eq. (5.6). The contribution to $\overline{|S_2|^2}$ in Eq. (5.4) that survives the large-t limit consists of the terms with coefficient $4t$, upon neglecting the exponentials $e^{-2t(1-\lambda^2)}$. This can be simplified to

$$\overline{|S_2|^2} \underset{t\to\infty}{\sim} 4t \int_{-1}^{+1} d\lambda \int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2 \left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^M e^{-2t(\lambda_2^2-1)} \frac{1}{\mathscr{R}} \cdot \frac{\lambda_2^2(\lambda_1 - \lambda\lambda_2)}{(1+\lambda)(\lambda_1+\lambda_2)}. \tag{C.4}$$

To extract the large-t limit completely, the procedure used above for $\overline{|S_1|^2}$ can be employed; namely, we write $\lambda_2 = 1 + z$ and expand in z, except for e^{-4tz}. This also yields half the integral expression of Eq. (5.6) and, hence, the result is established.

To now evaluate the GUE integral of Eq. (5.6) analytically, we first cast it in the equivalent form

$$\overline{|S_{\rm GUE}|^2} = 2\int_{-1}^{+1} d\lambda \int_1^\infty d\lambda_1 \left(\frac{1+\lambda}{1+\lambda_1}\right)^M \frac{1}{\lambda_1-\lambda} \cdot \frac{1}{(\lambda+1)(1+\lambda_1)}. \tag{C.5}$$

Then we make the change of variables

$$\lambda = \frac{1-w}{1+w},$$
$$\lambda_1 = \frac{1-w}{1+w} + \frac{2x}{1+w}. \tag{C.6}$$

The corresponding Jacobian is given by

$$\frac{\partial(\lambda, \lambda_1)}{\partial(x, w)} = \frac{4}{(1+w)^3}, \tag{C.7}$$

and we also note that

$$\frac{1}{\lambda_1 - \lambda} = \frac{1+w}{2x},$$
$$\frac{1}{(1+\lambda)(1+\lambda_1)} = \frac{(1+w)^2}{4(1+x)}, \tag{C.8}$$
$$\frac{1+\lambda}{1+\lambda_1} = \frac{1}{1+x}.$$

Putting all this together, we obtain

$$\overline{|S_{\rm GUE}|^2} = \int_0^{\infty} dx \int_0^{\infty} dw \, \frac{1}{x(1+x)^{M+1}} = \frac{1}{M}. \tag{C.9}$$

GOE Limit

To bring the integral in Eq. (5.8) into the form that appears in Ref. [14], we apply the transformation

$$\lambda = 1 - 2\mu,$$
$$\lambda_1 = [(1+\mu_1)(1+\mu_2) + \mu_1\mu_2 + 2\sqrt{\mu_1(1+\mu_1)\mu_2(1+\mu_2)}]^{1/2}, \tag{C.10}$$
$$\lambda_2 = [(1+\mu_1)(1+\mu_2) + \mu_1\mu_2 - 2\sqrt{\mu_1(1+\mu_1)\mu_2(1+\mu_2)}]^{1/2},$$

after re-expressing the non-compact integration in Eq. (5.8) as one over the triangular region $\lambda_2 \leqslant \lambda_1 < \infty$, $1 \leqslant \lambda_2 < \infty$, so that $\lambda_1 - \lambda_2 \geqslant 0$ everywhere. The associated Jacobian is given by

$$\frac{\partial(\lambda, \lambda_1, \lambda_2)}{\partial(\mu, \mu_1, \mu_2)} = \frac{\mu_1 - \mu_2}{\sqrt{\mu_1(1+\mu_1)\mu_2(1+\mu_2)}}, \tag{C.11}$$

and we note that

$$\mathscr{R} = 4(\mu + \mu_1)(\mu + \mu_2). \tag{C.12}$$

This allows us to write

$$\begin{aligned}\overline{|S_{ab}|^2} = \frac{1}{8}\int_0^1 d\mu \int_0^\infty d\mu_1 \int_0^\infty d\mu_2 \frac{|\mu_1 - \mu_2|}{\sqrt{\mu_1(1+\mu_1)\mu_2(1+\mu_2)}} \frac{(1-\mu)^M}{[(1+\mu_1)(1+\mu_2)]^{M/2}} \\ \times \frac{\mu(1-\mu)}{(\mu+\mu_1)^2(\mu+\mu_2)^2}\left[\frac{2}{1-\mu} - \frac{1}{1+\mu_1} - \frac{1}{1+\mu_2}\right],\end{aligned} \tag{C.13}$$

which agrees with Eq. (8.10) of Ref. [14] for $a \neq b$, after setting $T_c = 1$ for all c.

We now show how to evaluate the GOE integral exactly. With the help of the identity (D.18), the GOE limit of $\overline{|S_{ab}|^2}$, as given in Eq. (5.8), can be written as

$$\begin{aligned}\overline{|S_{\mathrm{GOE}}|^2} = 2\int_1^{+1} d\lambda \int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2 \left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^M \\ \times \frac{1-\lambda}{(\lambda_1+\lambda_2)^2}\frac{1}{\mathscr{R}}\left[1 + \frac{(1+\lambda)(\lambda_1\lambda_2 - \lambda)}{\mathscr{R}}\right].\end{aligned} \tag{C.14}$$

Further simplification results from observing that

$$\frac{\partial}{\partial\lambda}\left(\frac{1}{\mathscr{R}}\right) = \frac{2(\lambda_1\lambda_2 - \lambda)}{\mathscr{R}^2}, \tag{C.15}$$

in which case

$$\begin{aligned}\overline{|S_{\mathrm{GOE}}|^2} &= \int_{-1}^{+1} d\lambda \int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2 \left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^{M+2} \frac{1-\lambda}{1+\lambda}\left[\frac{2}{1+\lambda}\frac{1}{\mathscr{R}} + \frac{\partial}{\partial\lambda}\left(\frac{1}{\mathscr{R}}\right)\right] \\ &= \int_{-1}^{+1} d\lambda \frac{1}{1+\lambda}\left[1 - (M-1)\frac{1-\lambda}{1+\lambda}\right]\int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2 \left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^{M+2}\frac{1}{\mathscr{R}}\end{aligned} \tag{C.16}$$

after an integration by parts.

Next, we make the change of integration variables $(\lambda_1, \lambda_2) \mapsto (\bar{\lambda}, \alpha)$, where

$$\begin{aligned}\lambda_1 &= 1 + \tfrac{1}{2}(\bar{\lambda} - 1)(1 + \alpha), \\ \lambda_2 &= 1 + \tfrac{1}{2}(\bar{\lambda} - 1)(1 - \alpha),\end{aligned} \tag{C.17}$$

the associated Jacobian being

$$\frac{\partial(\lambda_1, \lambda_2)}{\partial(\bar{\lambda}, \alpha)} = \frac{1}{2}(\bar{\lambda} - 1). \tag{C.18}$$

Then

$$\overline{|S_{\mathrm{GOE}}|^2} = \frac{1}{2}\int_{-1}^{+1} d\alpha \int_{-1}^{+1} d\lambda \left[1 - (M-1)\frac{1-\lambda}{1+\lambda}\right] \int_1^{\infty} d\bar{\lambda} \left(\frac{1+\lambda}{1+\bar{\lambda}}\right)^{M+2} \frac{\bar{\lambda}-1}{\mathscr{R}}, \tag{C.19}$$

and we have

$$\mathscr{R} = (\bar{\lambda}-\lambda)^2 - \tfrac{1}{2}(1+\lambda)(\bar{\lambda}-1)^2(1-\alpha^2). \tag{C.20}$$

We are now in a position to implement the change of variables on the pair $(\lambda, \bar{\lambda})$ given in Eq. (C.6) (with λ_1 replaced by $\bar{\lambda}$ there), followed by setting $w = x(1-v)$. This leads to the representation

$$\overline{|S_{\mathrm{GOE}}|^2} = \int_0^{\infty} dx \frac{I_1(x)}{(1+x)^{M+2}} - (M-1)\int_0^{\infty} dx \frac{xI_2(x)}{(1+x)^{M+2}}, \tag{C.21}$$

where

$$\begin{aligned} I_1(x) &= \int_0^1 d\alpha \int_0^1 dv \frac{v}{(1-v)(x+1+v)+\alpha^2 v^2}, \\ I_2(x) &= \int_0^1 d\alpha \int_0^1 dv \frac{v(1-v)}{(1-v)(x+1+v)+\alpha^2 v^2}. \end{aligned} \tag{C.22}$$

The α-integration can be performed to yield the expressions

$$\begin{aligned} I_1(x) &= \int_0^1 dv \frac{1}{\sqrt{(1-v)(x+1+v)}} \arctan \frac{v}{\sqrt{(1-v)(x+1+v)}}, \\ I_2(x) &= \int_0^1 dv \sqrt{(1-v)/(x+1+v)} \arctan \frac{v}{\sqrt{(1-v)(x+1+v)}}. \end{aligned} \tag{C.23}$$

The integrals in Eqs. (C.23) are easily done for the special case of $x=0$ and give the results $I_1(0) = \pi^2/8$, $I_1(0) - I_2(0) = 1$. For general values of x, we can proceed by making the change of integration variable

$$v = \left(1 - \frac{x}{2}\right) - \frac{x}{2}u \tag{C.24}$$

and introducing the quantity

$$\beta = \frac{x/2}{1+x/2}. \tag{C.25}$$

Then we obtain

$$I_1(x) = \int_\beta^1 du \frac{1}{\sqrt{1-u^2}} \arctan \frac{u-\beta}{\sqrt{1-u^2}},$$
$$I_2(x) = \left(1+\frac{x}{2}\right) \int_\beta^1 du \frac{1-u}{\sqrt{1-u^2}} \arctan \frac{u-\beta}{\sqrt{1-u^2}}. \tag{C.26}$$

The derivatives with respect to β have simple forms:

$$\frac{dI_1}{d\beta} = -\frac{\ln(1+\beta)}{2\beta},$$
$$\frac{d\tilde{I}_2}{d\beta} = -\frac{1-\beta}{2\beta}\left[1-\frac{\ln(1+\beta)}{\beta}\right], \tag{C.27}$$

where we have introduced

$$\tilde{I}_2(x) \equiv I_2(x)/(1+x/2). \tag{C.28}$$

Now, after some integrations by parts, we can write

$$\overline{|S_{\mathrm{GOE}}|^2} = \frac{I_1(0)-I_2(0)}{M+1} + \frac{1}{M+1}\int_0^\infty dx \frac{I_1'(x)}{(1+x)^{M+1}}$$
$$-\frac{1}{2}\int_0^\infty dx \frac{\tilde{I}_2'(x)}{(1+x)^{M-1}} + \frac{M-1}{2(M+1)}\int_0^\infty dx \frac{\tilde{I}_2'(x)}{(1+x)^{M+1}}$$
$$= \frac{1}{M+1} + \int_0^1 d\beta \left(\frac{1-\beta}{1+\beta}\right)^M \frac{1}{1+\beta}\left[1-\frac{\ln(1+\beta)}{\beta}\right]$$
$$+ \frac{1}{2(M+1)}\int_0^1 \frac{d\beta}{\beta}\left(\frac{1-\beta}{1+\beta}\right)^{M+1}\left[1-\beta-\frac{\ln(1+\beta)}{\beta}\right]. \tag{C.29}$$

After noting that

$$\left(\frac{1-\beta}{1+\beta}\right)^M \frac{1}{(1+\beta)^2} = -\frac{1}{2(M+1)}\frac{d}{d\beta}\left(\frac{1-\beta}{1+\beta}\right)^{M+1} \tag{C.30}$$

and

$$\frac{d}{d\beta}\left\{(1+\beta)\left[1-\frac{\ln(1+\beta)}{\beta}\right]\right\} = -\frac{1}{\beta}\left[1-\beta-\frac{\ln(1+\beta)}{\beta}\right], \tag{C.31}$$

and using this to perform a further partial integration on the first integral on the RHS of Eq. (C.29), it becomes clear that the two integrals in Eq. (C.29) cancel each other, leaving us with the desired result of Eq. (5.8) in the main text.

Appendix D

Preliminaries

If we re-express $\mathscr{R}$ in terms of the variables (x, w, z) introduced in the previous appendix, and then make the further change of variables $(w, z) \mapsto (v', z')$, where $w = vx$, $z = 2xz'$, we obtain the form

$$\mathscr{R} = \frac{4x^2}{(1+vx)^2} D_v^x(z'), \tag{D.1}$$

with

$$D_v^x(z') \equiv (z' + 1 + vxz')^2 - 4(1-v)\, z'. \tag{D.2}$$

Now, in the large-t asymptotic expansion, due to the exponential factor $e^{-4tz} = e^{-8txz'}$, the significant domain of the integration is confined to values $xz' \ll 1$, and hence $vxz' \ll 1$ (since $0 \leqslant v \leqslant 1$). Since also $z' > 0$, we can always make the replacement $1 - vx'z \sim 1$ in our expression for $D_v^x(z')$, valid for large t. Thus, we have

$$D_v^x(z') \underset{t \gg 1}{\sim} (z'+1)^2 - 4(1-v)\, z' = D_v^0(z'). \tag{D.3}$$

We introduce the notation $D_v(z) \equiv D_v^0(z)$, and also note that one can write $D_v(z) = (z-1)^2 + 4vz$.

Likewise (in view of $xz' \ll 1$, $vxz' \ll 1$), we can make the following identifications and large-t replacements:

$$\lambda_1 \lambda_2 - \lambda = \frac{2x}{1+vx} [1 + (2-v)\, xz' + z']$$

$$\underset{t \gg 1}{\sim} \frac{2x}{1+vx} (1 + z'), \tag{D.4}$$

$$\lambda\lambda_2 - \lambda_1 = -\frac{2x}{1+vx} [1 + vxz' - z']$$

$$\underset{t \gg 1}{\sim} -\frac{2x}{1+vx} (1 - z'), \tag{D.5}$$

$$\lambda_2 - \lambda\lambda_1 = \frac{2x}{(1+vx)^2} [2v(1 + xz') + vx(1 + vxz') - 1 + z']$$

$$\underset{t \gg 1}{\sim} \frac{2x}{(1+vx)^2} (2v - 1 + vx + z'). \tag{D.6}$$

We should note that, in these combinations of the λ's, it is not justified to simply make the substitution $\lambda_2 = 1$ (i.e., $z' = 0$) in the asymptotic expansion, even when only leading order in t is sought, because it is not necessarily true that the significant integration region is confined to $z' \ll 1$.

The following result will be used extensively in the subsequent discussion: Suppose that $f(x)$ is a function such that $f(x \to \infty) \sim f_\infty(x) = Cx^\nu$. Then,

(i) if $\nu < -1$, we have

$$\int_0^\infty dx \frac{f(tx)}{(1+x)^{M+1}} \underset{t \gg 1}{\sim} \frac{1}{t} \int_0^\infty dx\, f(x); \tag{D.7}$$

(ii) if $\nu > -1$, we have

$$\int_0^\infty dx \frac{f(tx)}{(1+x)^{M+1}} \underset{t \gg 1}{\sim} t^\nu \int_0^\infty dx \frac{f_\infty(x)}{(1+x)^{M+1}}; \tag{D.8}$$

(iii) if $\nu = -1$, we have

$$\int_0^\infty dx \frac{f(tx)}{(1+x)^{M+1}} \underset{t \gg 1}{\sim} \frac{1}{t} \left\{ \int_0^1 dx\, f(x) + \int_0^\infty \left[f(x) - \frac{C}{x} \right] + C \left[\ln t - \sum_{j=1}^{M} \frac{1}{j} \right] \right\}. \tag{D.9}$$

It is useful to note that with $f(x \to \infty) \sim Cx^{-1}$ and $f(x)$ of the form

$$f(x) = \int_0^\infty dz\, e^{-8xz} g(z), \tag{D.10}$$

in which case we must have $g(0) = 8C$, Eq. (D.9) reads

$$\int_0^\infty dx \frac{f(tx)}{(1+x)^{M+1}} \underset{t \gg 1}{\sim} \frac{1}{8t} \left\{ \int_1^\infty dz \frac{g(z)}{z} + \int_0^1 \frac{g(z) - 8C}{z} + 8C \left[\ln 8t + \ln \gamma - \sum_{j=1}^{M} \frac{1}{j} \right] \right\}, \tag{D.11}$$

where γ denotes Euler's number.

It is also useful to define various functions for later convenience:

$$g_1(z) \equiv \int_0^1 dv \frac{1}{D_v(z)} = \frac{1}{4z} \ln \left(\frac{1+z}{1-z} \right)^2, \tag{D.12}$$

and we have $g_1(0) = 1$, $g_1(z \to \infty) \sim z^{-2}$;

$$g_2(z) \equiv \int_0^1 dv \frac{2v}{D_v(z)} = \frac{1}{2z} \left[1 - \frac{(1-z)^2}{4z} \ln \left(\frac{1+z}{1-z} \right)^2 \right], \tag{D.13}$$

and we have $g_2(0) = 1$, $g_2(z \to \infty) \sim z^{-2}$. Finally,

$$g_3(z) \equiv \int_0^1 dv \frac{2v}{D_v^2(z)} = \frac{1}{8z^2}\left[\ln\left(\frac{1+z}{1-z}\right)^2 + \left(\frac{1-z}{1+z}\right)^2 - 1\right]. \tag{D.14}$$

Here we have $g_3(0) = 1$, $g_3(z \to \infty) \sim z^{-4}$. Various integrals involving these functions can be established. These include

$$\int_0^\infty dz\, g_1(z) = \int_0^\infty dz\, g_2(z) = \frac{\pi^2}{4}, \qquad \int_0^\infty dz \frac{g_1(z) - 1}{z^2} = 0. \tag{D.15}$$

Also,

$$\int_0^\infty dz\, g_3(z) = 1, \qquad \int_0^1 dz \frac{g_3(z) - 1}{z} + \int_1^\infty dz \frac{g_3(z)}{z} = -\frac{1}{2}. \tag{D.16}$$

Large-t Expansion

We isolate from the full expression (5.2), four distinct contributions to the $O(t^{-1})$ term in $\overline{|S_1|^2}$:

$$\begin{aligned}
L_1 &= 4t \int_{-1}^{+1} d\lambda \int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2\, e^{-2t(\lambda_2^2-1)} \left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^{M+2} \frac{\lambda_2^2}{(1+\lambda)^3} \\
&\quad \times \left[1 + \frac{(1+\lambda)(\lambda_1\lambda_2 - \lambda)}{\mathscr{R}}\right], \\
A_1 &= \int_{-1}^{+1} d\lambda \int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2\, e^{-2t(\lambda_2^2-1)} \left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^{M+2} \frac{1}{(1+\lambda)^3} \left[1 + \frac{(1+\lambda)(\lambda_1\lambda_2 - \lambda)}{\mathscr{R}}\right] \\
&\quad \cdot \frac{1-\lambda^2-\lambda_1^2+\lambda_2^2}{\mathscr{R}}, \\
B_1 &= 4t \int_{-1}^{+1} d\lambda \int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2\, e^{-2t(\lambda_2^2-1)} \left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^{M+2} \frac{1}{(1+\lambda)^3} \\
&\quad \times \left[1 + \frac{(1+\lambda)(\lambda_1\lambda_2 - \lambda)}{\mathscr{R}}\right] (1-\lambda^2-\lambda_2^2)\, e^{-2t(1-\lambda^2)}, \\
C_1 &= \int_{-1}^{+1} d\lambda \int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2\, e^{-2t(\lambda_2^2-1)} \left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^{M+2} \frac{1}{(1+\lambda)^2} \frac{\lambda_1\lambda_2-\lambda}{\mathscr{R}^2} \\
&\quad \times (1-\lambda^2+\lambda_1^2-\lambda_2^2)\, e^{-2t(1-\lambda^2)}.
\end{aligned} \tag{D.17}$$

The quantity L_1 contains the term of leading order in t^{-1}, i.e., half the GUE limit, and it can be seen to be equivalent to the RHS of Eq. (C.1) upon making use of the identity

$$\frac{1-\lambda}{1+\lambda}+\frac{\lambda_1^2+\lambda_2^2-2}{(\lambda_1-\lambda_2)^2}=\frac{2\mathscr{R}}{(1+\lambda)(\lambda_1+\lambda_2)^2}\left[1+\frac{(1+\lambda)(\lambda_1\lambda_2-\lambda)}{\mathscr{R}}\right]. \tag{D.18}$$

We need to expand this term to first order in t^{-1}. The integrand of A_1 comprises the terms whose t-dependence resides in the common factor of $\exp[-2t(\lambda_2^2-1)]$. This contribution is non-leading, and its large-t limit will yield a term of order $O(t^{-1})$. The integrand of B_1 comprises the terms whose t-dependence resides in the common factor of $4t\exp[-2t(\lambda_2^2-1)]\exp[-2t(1-\lambda^2)]$. This contribution is also non-leading, and its large-t limit will yield a term of order $O(t^{-1})$. The term C_1 would appear to be of higher order in t^{-1}. However, because of the singular nature of the factor $\mathscr{R}^{-2}$, naive power counting does not apply, and C_1 does in fact contribute at order $O(t^{-1})$. There are four analogous contributions to the $O(t^{-1})$ term in the expression (5.4) for $\overline{|S_2|^2}$, given by

$$\begin{aligned}
L_2&=4t\int_{-1}^{+1}d\lambda\int_1^\infty d\lambda_1\int_1^\infty d\lambda_2\,e^{-2t(\lambda_2^2-1)}\left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^{M+1}\frac{\lambda_2^2}{(1+\lambda)^2}\frac{\lambda_1-\lambda\lambda_2}{\mathscr{R}},\\
A_2&=\int_{-1}^{+1}d\lambda\int_1^\infty d\lambda_1\int_1^\infty d\lambda_2\,e^{-2t(\lambda_2^2-1)}\left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^{M+1}\frac{1}{(1+\lambda)^2}\frac{1}{\mathscr{R}}\\
&\quad\times\left[4\lambda_2-\lambda_1-\lambda\lambda_2+\frac{2}{\mathscr{R}}(1-\lambda^2)(\lambda_1\lambda_2-\lambda)\,\lambda_2-\frac{2\lambda_2^2(\lambda_1+\lambda_2)}{1+\lambda}\right],\\
B_2&=4t\int_{-1}^{+1}d\lambda\int_1^\infty d\lambda_1\int_1^\infty d\lambda_2\,e^{-2t(\lambda_2^2-1)}\left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^{M+1}\frac{1}{(1+\lambda)^2}\frac{1}{\mathscr{R}}\\
&\quad\times[(\lambda_2-\lambda\lambda_1)(\lambda_2^2+\lambda^2-1)]\,e^{-2t(1-\lambda^2)},\\
C_2&=\int_{-1}^{+1}d\lambda\int_1^\infty d\lambda_1\int_1^\infty d\lambda_2\,e^{-2t(\lambda_2^2-1)}\left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^{M+1}\frac{1-\lambda}{1+\lambda}\frac{\lambda_1(\lambda_1\lambda_2-\lambda)}{\mathscr{R}^2}.
\end{aligned} \tag{D.19}$$

One deals with L_1 and L_2 most easily by combining them into $L=L_1+L_2$. Then

$$\begin{aligned}
L&=4t\int_{-1}^{+1}d\lambda\int_1^\infty d\lambda_1\int_1^\infty d\lambda_2\,e^{-2t(\lambda_2^2-1)}\left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^{M+2}\frac{\lambda_2^2}{(1+\lambda)^3}\\
&\quad+4t\int_{-1}^{+1}d\lambda\int_1^\infty d\lambda_1\int_1^\infty d\lambda_2\,e^{-2t(\lambda_2^2-1)}\left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^{M+2}\frac{\lambda_2^2}{(1+\lambda)^3}\frac{1}{\mathscr{R}}\\
&\quad\times[(\lambda_1-\lambda)^2+2(\lambda_2+\lambda)(\lambda_1-\lambda)-\lambda(\lambda_2-1)^2].
\end{aligned} \tag{D.20}$$

Next, we write $L = \overline{|S_{\mathrm{GUE}}|^2} + \delta L$ which isolates the part that is zeroth order in t^{-1}. To extract δL, which denotes the contribution of order $O(t^{-1})$, we must expand the integrand, save for e^{-4tz}, to first or second order in $z = \lambda_2 - 1$. We shall split the result of the expansion into three components, $\delta L = \delta L' + \delta L'' + \delta L'''$, where $\delta L'$ represents the contribution from the first term on the RHS of Eq. (D.20), $\delta L''$ is the contribution coming from the expression in square brackets in the second term on the RHS of Eq. (D.20), and $\delta L'''$ denotes the rest. Then we have

$$\begin{aligned}\delta L' &= 4t \int_{-1}^{+1} d\lambda \int_1^{\infty} d\lambda_1 \left(\frac{1+\lambda}{1+\lambda_1}\right)^{M+2} \frac{1}{(1+\lambda)^3} \int_0^{\infty} dz\, e^{-4tz} \\ &\quad \times \left[-2tz^2 + 2z - (M+2)\frac{z}{1+\lambda_1}\right] \\ &= \frac{1}{16t}\left[2\int_0^{\infty} dx \frac{x}{(1+x)^{M+2}} \int_0^1 dv - (M+2)\int_0^{\infty} dx \frac{x}{(1+x)^{M+3}} \int_0^1 dv\,(1+vx)\right] \\ &= \frac{1}{16t}\left[\frac{1}{M} - \frac{2}{M+1}\right]. \end{aligned} \tag{D.21}$$

Next,

$$\begin{aligned}\delta L'' &= 4t \int_{-1}^{+1} d\lambda \int_1^{\infty} d\lambda_1 \int_0^{\infty} dz\, e^{-4tz} \left(\frac{1+\lambda}{1+\lambda_1}\right)^{M+2} \frac{1}{(1+\lambda)^3} \frac{2z(\lambda_1-\lambda) - z^2\lambda}{\mathscr{R}} \\ &\underset{t\gg 1}{\sim} 4t \int_0^{\infty} dx \frac{x^2}{(1+x)^{M+2}} \int_0^1 dv\,(1+vx) \int_0^{\infty} dz\, e^{-8tvz} \frac{2z - z^2}{D_v(z)}. \end{aligned} \tag{D.22}$$

The contribution involving z^2 is non-leading, being of order $O(\ln t/t^2)$, while in the rest we can set $D_v(z) \sim D_v(0) = 1$ by virtue of Eq. (D.7), to obtain

$$\begin{aligned}\delta L'' &\underset{t\gg 1}{\sim} \frac{1}{8t}\int_0^{\infty} dx \frac{1}{(1+x)^{M+2}} \int_0^1 dv\,(1+vx) \\ &= \frac{1}{16t}\left[\frac{1}{M} + \frac{1}{M+1}\right]. \end{aligned} \tag{D.23}$$

It is convenient to split $\delta L'''$ into a sum of two terms $\delta L''' = \delta L_1''' + \delta L_2'''$. For the first term, we write

$$\begin{aligned}\delta L_1''' &= 4t \int_{-1}^{+1} d\lambda \int_1^{\infty} d\lambda_1 \int_0^{\infty} dz\, e^{-4tz} \left(\frac{1+\lambda}{1+\lambda_1}\right)^{M+2} \frac{1}{(1+\lambda)^3} \frac{1}{\mathscr{R}} \\ &\quad \times \left[-2tz^2 + 2z - (M+2)\frac{z}{1+\lambda_1}\right] [(\lambda_1-\lambda)^2 + 2(1+\lambda)(\lambda_1-\lambda)]. \end{aligned} \tag{D.24}$$

Due to the presence of the factor $\lambda_1 - \lambda$ in the numerator, it is safe here to set $\mathscr{R} \sim_{z \to 0} (\lambda_1 - \lambda)^2$ for large t. This results in

$$\delta L_1''' = \frac{1}{4t}\int_{-1}^{+1} d\lambda \int_1^{\infty} d\lambda_1 \left(\frac{1+\lambda}{1+\lambda_1}\right)^{M+2} \frac{1}{(1+\lambda)^3}\left[1+\frac{2(1+\lambda)}{\lambda_1-\lambda}\right]\left[1-\frac{M+2}{1+\lambda_1}\right]$$

$$= \frac{1}{16t}\int_0^{\infty} dx \frac{x+2}{(1+x)^{M+3}}\int_0^1 dv\,[2(1+x)-(M+2)(1+vx)]$$

$$= \frac{1}{16t}\left[-2+\frac{1}{M}+\frac{1}{M+1}\right]. \tag{D.25}$$

The remaining contribution to $\delta L'''$ is given by

$$\delta L_2''' = 4t\int_{-1}^{+1} d\lambda \int_1^{\infty} d\lambda_1 \int_0^{\infty} dz\, e^{-4tz}\left(\frac{1+\lambda}{1+\lambda_1}\right)^{M+2}\frac{1}{(1+\lambda)^3}[(\lambda_1-\lambda)^2$$

$$+2(1+\lambda)(\lambda_1-\lambda)]\left[\frac{1}{\mathscr{R}}-\frac{1}{(\lambda_1-\lambda)^2}\right]. \tag{D.26}$$

Using the relation

$$\frac{(\lambda_1-\lambda)^2}{\mathscr{R}} - 1 \underset{t \gg 1}{\sim} \frac{D_v(z') - 1 + 2vxz'}{D_v(z')}, \tag{D.27}$$

we find

$$\delta L_2''' \underset{t \gg 1}{\sim} 4t\int_0^{\infty} dx \frac{x(x+2)}{(1+x)^{M+2}}\int_0^1 dv \int_0^{\infty} dz\, e^{-8txz}\left[\left(\frac{1}{D_v(z)}-1\right)-\frac{2vxz}{D_v(z)}\right]$$

$$= \delta L_{2a}''' + \delta L_{2b}''', \tag{D.28}$$

where the two components pick out each of the two terms in the square brackets, respectively. In $\delta L_{2b}'''$, we are allowed by Eq. (D.8) to set $D_v(z) \sim D_v(0) = 1$, and so we obtain

$$\delta L_{2b}''' \underset{t \gg 1}{\sim} -\frac{1}{8t}\int_0^{\infty} dx \frac{x+2}{(1+x)^{M+2}}\int_0^1 dv\, v$$

$$= -\frac{1}{16t}\left[\frac{1}{M}+\frac{1}{M+1}\right]. \tag{D.29}$$

The component $\delta L_{2a}'''$ can be expressed as

$$\delta L_{2a}''' = \int_0^{\infty} dx \frac{f(tx)}{(1+x)^{M+1}} + \int_0^{\infty} dx \frac{f(tx)}{(1+x)^{M+2}}, \tag{D.30}$$

where

$$f(x) = 4x \int_0^\infty dz\, e^{-8xz}[g_1(z) - 1] \underset{x \to \infty}{\sim} -\frac{1}{3}\frac{1}{(8x)^2}. \qquad \text{(D.31)}$$

In this case, Eq. (D.7) implies that

$$\delta L'''_{2a} \underset{t \gg 1}{\sim} \frac{2}{t}\int_0^\infty dx\, f(x) = -\frac{1}{8t}\int_0^\infty dz \frac{g_1(z) - 1}{z^2}. \qquad \text{(D.32)}$$

We see from Eq. (D.15) that $\delta L'''_{2a}$ vanishes up to order $O(t^{-1})$, and consequently only $\delta L'''_{2b}$ survives in $\delta L'''_2$. Collecting together Eqs. (D.21), (D.23) (D.25), and (D.29), we obtain

$$\delta L \underset{t \gg 1}{\sim} \frac{1}{16t}\left[-2 + \frac{2}{M} - \frac{1}{M+1}\right]. \qquad \text{(D.33)}$$

To extract the large-t limit of A_1, we proceed as above by writing $\lambda_2 = 1 + z$ and expanding the integrand to lowest order in z, except for e^{-4tz} and the combinations in Eqs. (D.4)–(D.5) which are kept intact. But first we re-express A_1 as

$$A_1 = \int_{-1}^{+1} d\lambda \int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2\, e^{-2t(\lambda_2^2 - 1)} \left(\frac{1+\lambda}{\lambda_1 + \lambda_2}\right)^{M+2} \frac{1}{(1+\lambda)^3}\left[\frac{(1+\lambda)(\lambda_1\lambda_2 - \lambda)}{\mathscr{R}} + 1\right]$$
$$\times \left[\frac{2\lambda_2(\lambda_2 - \lambda\lambda_1)}{\mathscr{R}} - 1\right]. \qquad \text{(D.34)}$$

It is now convenient to write $A_1 = A_1^{(0)} + A_1^{(1)} + A_1^{(2)}$, where $A_1^{(n)}$ denotes the contribution to A_1, as given above, involving the factor $1/\mathscr{R}^n$. So, we have

$$A_1^{(0)} \underset{t \gg 1}{\sim} -\int_{-1}^{+1} d\lambda \int_1^\infty d\lambda_1 \int_0^\infty dz\, e^{-4tz}\left(\frac{1+\lambda}{1+\lambda_1}\right)^{M+2} \frac{1}{(1+\lambda)^3}$$
$$= -\frac{1}{8t}\int_0^\infty \frac{x}{(1+x)^{M+2}} \int_0^1 dv$$
$$= -\frac{1}{8t}\left[\frac{1}{M} - \frac{1}{M+1}\right]. \qquad \text{(D.35)}$$

Next, if we note that by Eqs. (D.4) and (D.6) we can make the replacement

$$2\lambda_2(\lambda_2 - \lambda\lambda_1) - (1+\lambda)(\lambda_1\lambda_2 - \lambda) \underset{t \gg 1}{\sim} \frac{4x}{(1+vx)^2}[2(v-1) + vx], \qquad \text{(D.36)}$$

then we have

$$A_1^{(1)} \underset{t \gg 1}{\sim} -\int_{-1}^{+1} d\lambda \int_0^{\infty} d\lambda_1 \int_0^{\infty} dz\, e^{-4tz} \left(\frac{1+\lambda}{1+\lambda_1}\right)^{M+2} \frac{1}{(1+\lambda)^3} \frac{1}{\mathscr{R}}$$

$$\times [2\lambda_2(\lambda_2 - \lambda\lambda_1) - (1+\lambda)(\lambda_1\lambda_2 - \lambda)]$$

$$\underset{t \gg 1}{\sim} \int_0^{\infty} dx \frac{x}{(1+x)^{M+2}} \int_0^1 dv \int_0^{\infty} dz\, e^{-8txz} \frac{2(v-1)+vx}{D_v(z)}$$

$$\underset{t \gg 1}{\sim} \int_0^{\infty} dx \frac{1}{(1+x)^{M+2}} \int_0^1 dv\, [2(v-1)+vx]$$

$$= \frac{1}{16t}\left[\frac{1}{M} - \frac{3}{M+1}\right]. \tag{D.37}$$

The observation that

$$\frac{2(\lambda_2 - \lambda\lambda_1)}{\mathscr{R}} = -\frac{\partial}{\partial\lambda_2}\left(\frac{1}{\mathscr{R}}\right) \tag{D.38}$$

allows us to express $A_1^{(2)}$ in a form that is amenable to integration by parts. It is convenient to do so in order to avoid the emergence of singular integrals (convergent only as principal values) in the process of reducing to the large-t limit. Accordingly, we write

$$A_1^{(2)} \underset{t \gg 1}{\sim} -\int_{-1}^{+1} d\lambda \int_1^{\infty} d\lambda_1 \int_0^{\infty} dz\, e^{-4tz} \left(\frac{1+\lambda}{1+\lambda_1}\right)^{M+2} \frac{1}{(1+\lambda)^2} (\lambda_1\lambda_2 - \lambda) \frac{\partial}{\partial z}\left(\frac{1}{\mathscr{R}}\right)$$

$$\underset{t \gg 1}{\sim} -\int_{-1}^{+1} d\lambda \int_1^{\infty} d\lambda_1 \int_0^{\infty} dz\, e^{-4tz} \left(\frac{1+\lambda}{1+\lambda_1}\right)^{M+2} \frac{1}{(1+\lambda)^2}$$

$$\times \left\{4t\left[\frac{\lambda_1 - \lambda}{\mathscr{R}} - \frac{1}{\lambda_1 - \lambda}\right] - (1-4tz)\frac{\lambda_1}{\mathscr{R}}\right\}$$

$$= A_1^{(2a)} + A_1^{(2b)}. \tag{D.39}$$

Let us first deal with $A_1^{(2b)}$. We can write this component as

$$A_1^{(2b)} = \left(1 + t\frac{d}{dt}\right) \tilde{A}_1^{(2b)} \tag{D.40}$$

with

$$\tilde{A}_1^{(2b)} = \int_{-1}^{+1} d\lambda \int_1^{\infty} d\lambda_1 \int_0^{\infty} dz\, e^{-4tz} \left(\frac{1+\lambda}{1+\lambda_1}\right)^{M+2} \frac{\lambda_1}{(1+\lambda)^2} \frac{1}{\mathscr{R}}$$

$$\underset{t \gg 1}{\sim} \frac{1}{2}\int_0^{\infty} dx \frac{1}{(1+x)^{M+2}} \int_0^1 dv\, [(2-v)x+1] \int_0^{\infty} dz\, e^{-8txz} \frac{1}{D_v(z)}. \tag{D.41}$$

Now, the leading contribution to the term involving $(2-v)$, which is of order $O(t^{-1})$, is annihilated by the differential operator in Eq. (D.40). In the rest, only the logarithmic term, whose presence follows from Eq. (D.9), survives. Thus, we obtain

$$\begin{aligned} A_1^{(2b)} &\underset{t\gg 1}{\sim} \frac{1}{2}\left(1+t\frac{d}{dt}\right)\int_0^\infty dx\,\frac{1}{(1+x)^{M+2}}\int_0^\infty dz\,e^{-8txz}g_1(z) \\ &\underset{t\gg 1}{\sim} \frac{1}{16}\left(1+t\frac{d}{dt}\right)\frac{\ln t}{t} \\ &= \frac{1}{16t}. \end{aligned} \tag{D.42}$$

Finally, we have

$$A_1^{(2a)} \underset{t\gg 1}{\sim} -4t\int_{-1}^{+1} d\lambda\int_1^\infty d\lambda_1\int_0^\infty dz\,e^{-4tz}\left(\frac{1+\lambda}{1+\lambda_1}\right)^{M+2}\frac{\lambda_1-\lambda}{(1+\lambda)^2}\left[\frac{1}{\mathscr{R}}-\frac{1}{(\lambda_1-\lambda)^2}\right]. \tag{D.43}$$

With the help of Eq. (D.27), this can be reduced to the form

$$A_1^{(2a)} \underset{t\gg 1}{\sim} -4t\int_0^\infty dx\,\frac{x}{(1+x)^{M+2}}\int_0^1 dv\int_0^\infty dz\,e^{-8tzx}\left[\left(\frac{1}{D_v(z)}-1\right)-\frac{2xvz}{D_v(z)}\right]. \tag{D.44}$$

One should note that this expression for $A_1^{(2a)}$ has a structure very similar to that of $\delta L_2'''$ in Eq. (D.28), and an almost identical analysis leads to the result

$$A_1^{(2a)} \underset{t\gg 1}{\sim} \frac{1}{16t}\frac{1}{M+1}. \tag{D.45}$$

Adding together all the contributions to $A_1^{(2)}$ and combining this with the results (D.35) for $A_1^{(0)}$ and (D.37) for $A_1^{(1)}$, we see that

$$A_1 \underset{t\gg 1}{\sim} \frac{1}{16t}\left(1-\frac{1}{M}\right). \tag{D.46}$$

To extract the large-t limit of A_2, we also split it into a sum of regular and singular parts $A_2 = A_2^{(1)} + A_2^{(2)}$, where $A_2^{(n)}$ denotes the contribution to A_2 in Eq. (D.19) that involves $1/\mathscr{R}^n$. Then

$$\begin{aligned} A_2^{(1)} &= \int_{-1}^{+1} d\lambda\int_1^\infty d\lambda_1\int_1^\infty d\lambda_2\,e^{-2t(\lambda_2^2-1)}\left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^{M+1}\frac{1}{(1+\lambda)^3}\frac{1}{\mathscr{R}} \\ &\quad\times\{2\lambda_2[(1-\lambda^2)-(\lambda_1-\lambda)]+(1+\lambda)(\lambda_1\lambda_2-\lambda)+2\lambda_2(1-\lambda_2)\} \\ &= A_2^{(1a)}+A_2^{(1b)}+A_2^{(1c)}, \end{aligned} \tag{D.47}$$

where the three components pick out each of the three terms in the braces, respectively. The first component yields

$$A_2^{(1a)} \underset{t \gg 1}{\sim} 2\int_{-1}^{+1} d\lambda \int_1^{\infty} d\lambda_1 \int_0^{\infty} dz\, e^{-4tz} \left(\frac{1+\lambda}{1+\lambda_1}\right)^{M+1} \frac{1}{(1+\lambda)^3} \frac{(1-\lambda^2)-(\lambda_1-\lambda)}{\mathscr{R}}$$

$$\underset{t \gg 1}{\sim} \int_0^{\infty} dx \frac{x}{(1+x)^{M+1}} \int_0^1 dv\, [(2v-1)+vx] \int_0^{\infty} dz\, e^{-8txz} \frac{1}{D_v(z)}$$

$$\underset{t \gg 1}{\sim} -\frac{1}{16t}\left[\frac{1}{M-1}-\frac{1}{M}\right], \tag{D.48}$$

having set $D_v(z) \sim D_v(0) = 1$ by virtue of Eq. (D.8). With the aid of Eq. (D.5), the second component gives us

$$A_2^{(1b)} \underset{t \gg 1}{\sim} \int_1^{+1} d\lambda \int_1^{\infty} d\lambda_1 \int_0^{\infty} dz\, e^{-4tz} \left(\frac{1+\lambda}{1+\lambda_1}\right)^{M+1} \frac{1}{(\lambda+1)^2} \frac{\lambda\lambda_2-\lambda_1}{\mathscr{R}}$$

$$\underset{t \gg 1}{\sim} -\int_0^{\infty} dx \frac{x}{(1+x)^{M+1}} \int_0^{\infty} dz\, e^{-8txz}(1-z) \int_0^1 dv \frac{1}{D_v(z)}. \tag{D.49}$$

The term involving z is non-leading, being of order $O(\ln t/t^2)$, while in the rest we can set $D_v(z) \sim D_v(0) = 1$ to obtain

$$A_2^{(1b)} \underset{t \gg 1}{\sim} -\frac{1}{8t}\frac{1}{M}. \tag{D.50}$$

The contribution from the third component $A_2^{(1c)}$ is of higher order $O(\ln t/t^2)$ and, hence, it is discarded. On combining the three components, we obtain

$$A_2^{(1)} \underset{t \gg 1}{\sim} -\frac{1}{16t}\left[\frac{1}{M}+\frac{1}{1+M}\right]. \tag{D.51}$$

Taking account of Eq. (D.4), we have for $A_2^{(2)}$,

$$A_2^{(2)} \underset{t \gg 1}{\sim} 2\int_{-1}^{+1} d\lambda \int_1^{\infty} d\lambda_1 \int_0^{\infty} dz\, e^{-4tz} \left(\frac{1+\lambda}{1+\lambda_1}\right)^{M+1} \frac{1-\lambda}{1+\lambda} \frac{\lambda_1\lambda_2-\lambda}{\mathscr{R}^2}$$

$$\underset{t \gg 1}{\sim} \int_0^{\infty} dx \frac{1}{(1+x)^{M+1}} \int_0^{\infty} dz\, e^{-8txz}(1+z) \int_0^1 dv \frac{2v}{D_v^2(z)}. \tag{D.52}$$

We express this as

$$A_2^{(2)} \underset{t \gg 1}{\sim} \int_0^{\infty} dx \frac{f_1(tx)}{(1+x)^{M+1}} + \int_0^{\infty} dx \frac{f_2(tx)}{(1+x)^{M+1}}, \tag{D.53}$$

where

$$f_1(x) = \int_0^\infty dz\, e^{-8xz} g_3(z) \underset{x\to\infty}{\sim} \frac{1}{8x},$$
$$f_2(x) = \int_0^\infty dz\, e^{-8xz} z g_3(z) \underset{x\to\infty}{\sim} \frac{1}{(8x)^2}. \tag{D.54}$$

Then, Eq. (D.7) applied to $f_2(x)$ and Eq. (D.11) applied to $f_1(x)$, together with the integral identities (D.16), imply that

$$A_2^{(2)} \underset{t\gg 1}{\sim} \frac{1}{8t}\left[\frac{1}{2} + \ln 8t + \ln\gamma - \sum_{j=1}^{M}\frac{1}{j}\right]. \tag{D.55}$$

The full result for A_2 is obtained by adding Eqs. (D.51) and (D.55).

The sum $B = B_1 + B_2$ can be simplified to yield

$$B = 4t\int_{-1}^{+1} d\lambda \int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2\, e^{-2t(\lambda_2^2-1)} e^{-2t(1-\lambda^2)} \left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^{M+2} \frac{\lambda_1^2-1}{(1+\lambda)^2}\frac{1-\lambda^2-\lambda_2^2}{\mathscr{R}}$$
$$\underset{t\gg 1}{\sim} -2t\int_{-1}^{+1} d\lambda \int_{\lambda}^\infty d\lambda_1 \int_0^\infty dz\, e^{-4tz} e^{-2t(1-\lambda^2)} \left(\frac{1+\lambda}{1+\lambda_1}\right)^{M+1} \frac{\lambda_1-1}{(1+\lambda)^2}\frac{1}{\mathscr{R}}$$
$$\underset{t\gg 1}{\sim} -8t\int_0^\infty dx \frac{x}{(1+x)^{M+1}} \int_0^1 dv\, e^{-8txv}(1-v) \int_0^\infty dz\, e^{-8txz} \frac{1}{D_v(z)}$$
$$= \sum_{\alpha=0}^{2} \int_0^\infty dx \frac{f_\alpha(tx)}{(1+x)^{M+1}}, \tag{D.56}$$

where we have used the fact that only values $w = vx \ll 1$ are significant for large t, and we have found it convenient to introduce

$$f_0(x) = -\int_0^\infty dv\, e^{-8xv} \underset{x\to\infty}{\sim} -\frac{1}{8x},$$
$$f_1(x) = -8x\int_0^1 dv\, e^{-8xv} \int_0^\infty dz\, e^{-8xz}\left[\frac{1}{D_v(z)} - 1\right] \underset{x\to\infty}{\sim} -\frac{2}{(8x)^2}, \tag{D.57}$$
$$f_2(x) = 8x\int_0^1 dv\, v e^{-8xv} \int_0^\infty dz\, e^{-8xz} \frac{1}{D_v(z)} \underset{x\to\infty}{\sim} \frac{1}{(8x)^2}.$$

Application of Eq. (D.7) to $f_1(x)$ and $f_2(x)$, followed by the x-integration, results in

$$\int_0^\infty dx \frac{f_1(tx)}{(1+x)^{M+1}} \underset{t\gg 1}{\sim} -\frac{1}{8t}\int_0^1 dv \int_0^\infty dz \frac{1}{(v+z)^2}\left[\frac{1}{D_v(z)} - 1\right] = -\frac{1}{16t}\int_0^\infty dz\, g_2(z),$$
$$\int_0^\infty dx \frac{f_2(tx)}{(1+x)^{M+1}} \underset{t\gg 1}{\sim} \frac{1}{8t}\int_0^1 dv \int_0^\infty dz \frac{v}{(v+z)^2}\frac{1}{D_v(z)} = \frac{1}{16t}\int_0^\infty dz\, g_2(z). \tag{D.58}$$

These two contributions cancel out and leave us with

$$B \underset{t \gg 1}{\simeq} \int_0^{\infty} dx \frac{f_0(tx)}{(1+x)^{M+1}} \underset{t \gg 1}{\simeq} -\frac{1}{8t}\left[\ln 8t + \ln \gamma - \sum_{j=1}^{M} \frac{1}{j}\right], \tag{D.59}$$

according to Eq. (D.11), having noted that in this application $g(z) = \Theta(1-z)$, with Θ denoting the Heaviside step function. We note that B cancels the non-analytic term in A_2.

We combine $C = C_1 + C_2$ and note that $1 - \lambda^2 + \lambda_1^2 - \lambda_2^2 = -\mathscr{R} + 2\lambda_1(\lambda_1 - \lambda\lambda_2)$. Then we decompose $C = C^{(1)} + C^{(2)}$ according to the power of $\mathscr{R}^{-1}$ that the contributions involve. Thus, we have

$$C^{(1)} = \int_{-1}^{+1} d\lambda \int_1^{\infty} d\lambda_1 \int_1^{\infty} d\lambda_2\, e^{-2t(\lambda_2^2-1)} e^{-2t(1-\lambda^2)} \left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^{M+2} \times \frac{1}{(1+\lambda)^2} \frac{\lambda_1\lambda_2 - \lambda}{\mathscr{R}}, \tag{D.60}$$

$$C^{(2)} = \int_{-1}^{+1} d\lambda \int_1^{\infty} d\lambda_1 \int_1^{\infty} d\lambda_2\, e^{-2t(\lambda_2^2-1)} e^{-2t(1-\lambda^2)} \left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^{M+2} \times \frac{\lambda_1}{(1+\lambda)^2} (\lambda_1\lambda_2 - \lambda) \frac{\partial}{\partial\lambda_2}\left(\frac{1}{\mathscr{R}}\right). \tag{D.61}$$

The term $C^{(1)}$ is clearly of higher order in $1/t$. To evaluate $C^{(2)}$, we first integrate by parts in order to avoid the appearance of singular integrals, as in $A_1^{(2)}$. This yields

$$\begin{aligned} C^{(2)} &\underset{t \gg 1}{\simeq} \int_{-1}^{+1} d\lambda \int_1^{\infty} d\lambda_1 \int_0^{\infty} dz\, e^{-4tz} e^{-2t(1-\lambda^2)} \left(\frac{1+\lambda}{1+\lambda_1}\right)^{M+2} \\ &\quad \times \frac{\lambda_1}{(1+\lambda)^2} (\lambda_1\lambda_2 - \lambda) \frac{\partial}{\partial z}\left(\frac{1}{\mathscr{R}}\right) \\ &\underset{t \gg 1}{\simeq} \int_{-1}^{+1} d\lambda \int_1^{\infty} d\lambda_1 \int_0^{\infty} dz\, e^{-4tz} e^{-2t(1-\lambda^2)} \\ &\quad \times \left(\frac{1+\lambda}{1+\lambda_1}\right)^{M+2} \frac{\lambda_1}{(1+\lambda)^2} \left\{4t\left[\frac{\lambda_1-\lambda}{\mathscr{R}} - \frac{1}{\lambda_1-\lambda}\right] - (1-4tz)\frac{\lambda_1}{\mathscr{R}}\right\} \\ &= C^{(2a)} + C^{(2b)}. \end{aligned} \tag{D.62}$$

Making use of Eq. (D.27) gives us

$$\begin{aligned} C^{(2a)} &\underset{t \gg 1}{\simeq} 4t \int_0^{\infty} dx \frac{x(1+2x)}{(1+x)^{M+2}} \int_0^1 dv\, e^{-8txv} \int_0^{\infty} dz\, e^{-8txz} \left[\frac{1}{D_v(z)} - 1\right] \\ &\underset{t \gg 1}{\simeq} \int_0^{\infty} dx \frac{f(tx)}{(1+x)^{M+1}} - \frac{1}{2}\int_0^{\infty} dx \frac{f(tx)}{(1+x)^{M+2}}, \end{aligned} \tag{D.63}$$

with

$$f(x)=8x\int_0^1 dv\, e^{-8xv}\int_0^\infty dz\, e^{-8xz}\left[\frac{1}{D_v(z)}-1\right]\underset{x\to\infty}{\sim}\frac{2}{(8x)^2}. \tag{D.64}$$

Then, using Eq. (D.7) and performing the x-integration, we have

$$C^{(2a)}\underset{t\gg 1}{\sim}\frac{1}{16t}\int_0^1 dv\int_0^\infty dz\frac{1}{(v+z)^2}\left[\frac{1}{D_v(z)}-1\right]=\frac{1}{32t}\int_0^\infty dz\, g_2(z). \tag{D.65}$$

For the component $C^{(2b)}$, we can write

$$C^{(2b)}\underset{t\gg 1}{\sim}\frac{1}{2}\int_0^\infty dx\left[\frac{1}{(1+x)^{M+2}}+\frac{4x}{(1+x)^{M+1}}\right]\int_0^1 dv\, e^{-8txv}$$
$$\times\int_0^\infty dz\,(1-8txz)\, e^{-8txz}\frac{1}{D_v(z)}. \tag{D.66}$$

The second term in the square brackets is non-leading, leaving us with

$$C^{(2b)}\underset{t\gg 1}{\sim}\int_0^\infty dx\frac{f(tx)}{(1+x)^{M+2}}, \tag{D.67}$$

where

$$f(x)=-\frac{1}{2}\int_0^1 dv\, e^{-8xv}\int_0^\infty dz\,(1-8xz)\, e^{-8xz}\frac{1}{D_v(z)}\underset{x\to\infty}{\sim}\frac{1}{(8x)^3}. \tag{D.68}$$

Hence, after appealing to Eq. (D.7) and carrying out the x-integration, we obtain

$$C^{(2b)}\underset{t\gg 1}{\sim}-\frac{1}{16t}\int_0^1 dv\int_0^\infty dz\frac{v}{(v+z)^2}\frac{1}{D_v(z)}=-\frac{1}{32t}\int_0^\infty dz\, g_2(z). \tag{D.69}$$

We see from Eqs. (D.65) and (D.69) that $C^{(2)}$ vanishes up to order $O(t^{-1})$, and therefore C provides no contribution to the final result. All the separate components can now be combined to give

$$\delta L+\sum_{j=1}^{2}(A_j+B_j+C_j)\underset{t\gg 1}{\sim}-\frac{1}{8t}\cdot\frac{M}{M^2-1}, \tag{D.70}$$

which leads directly to Eqs. (5.10) and (5.11) in the main text.

Additional Remarks

The double integrals which appear in Eqs. (D.58), (D.65), and (D.69) can be easily performed by making the change of integration variables $(v,z)\mapsto(\tilde v,\tilde z)$ defined by

$$v=\frac{\tilde v\tilde z}{1+\tilde v\tilde z},\qquad z=\frac{(1-\tilde v)\,\tilde z}{1+\tilde v\tilde z}. \tag{D.71}$$

Then we have

$$\int_0^1 dv \int_0^\infty dz = \int_0^1 d\tilde{v} \int_0^\infty d\tilde{z} \frac{\tilde{z}}{(1+\tilde{v}\tilde{z})^3},$$
$$D_v(z) = \frac{D_{\tilde{v}}(\tilde{z})}{(1+\tilde{v}\tilde{z})^2}, \tag{D.72}$$
$$\frac{1-v}{v+z} = \frac{1}{\tilde{z}}.$$

The result used in the text follow immediately.

Appendix E

In this appendix, we outline the expansion of $\overline{|S_{ab}|^2}$ in the limit of large M and t, subject to fixed ratio t/M. With the aid of Eq. (D.18), $\overline{|S_{ab}|^2}$ can be brought into the form

$$\overline{|S_{ab}|^2} = \frac{1}{2}\int_{-1}^{+1} d\lambda \int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2 \left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^M e^{-2t(\lambda_2^2-1)}$$
$$\times\left\{p_0 + \frac{p_1}{\mathscr{R}} + \frac{2(\lambda_1\lambda_2-\lambda)\,p_2}{\mathscr{R}^2} + \left[q_0 + \frac{q_1}{\mathscr{R}} + \frac{2(\lambda_1\lambda_2-\lambda)\,q_2}{\mathscr{R}^2}\right] e^{-2t(1-\lambda^2)}\right\}, \tag{E.1}$$

where the p_i and q_i are rational functions of λ, λ_1, and λ_2. After using

$$\frac{\partial}{\partial\lambda}\left(\frac{1}{\mathscr{R}}\right) = \frac{2(\lambda_1\lambda_2-\lambda)}{\mathscr{R}^2}, \tag{E.2}$$

we integrate by parts with respect to λ and obtain

$$\overline{|S_{ab}|^2} = \frac{1}{2}\int_{-1}^{+1} d\lambda \int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2 \left(\frac{1+\lambda}{\lambda_1+\lambda_2}\right)^M \frac{1}{\mathscr{R}}\left[\tilde{p}e^{-2t(\lambda_2^2-1)} + \tilde{q}e^{-2t(\lambda_2^2-\lambda^2)}\right], \tag{E.3}$$

where

$$\tilde{p} = \mathscr{R}p_0 + p_1 - \frac{\partial p_2}{\partial\lambda} - \frac{M}{1+\lambda}p_2,$$
$$\tilde{q} = \mathscr{R}q_0 + q_1 - \frac{\partial q_2}{\partial\lambda} - \frac{M}{1+\lambda}q_2 - 4t\lambda q_2. \tag{E.4}$$

The surface term arising from the partial integration vanishes, since

$$p_2(\lambda=0) + q_2(\lambda=0) = 0, \tag{E.5}$$

even for $\overline{|S_1|^2}$ and $\overline{|S_2|^2}$ separately.

In order to obtain the integral in the limit $M \to \infty$ and $t \to \infty$ for fixed ratio t/M, we use the fact that in this limit, the main contribution to the integrals comes from the region $\lambda, \lambda_1, \lambda_2 \approx 1$. Therefore, it is convenient to introduce a scalar factor r and new variables of integration x, x_1, x_2, according to

$$t = t'/r, \qquad M = M'/r, \tag{E.6}$$

and

$$\begin{aligned} \lambda &= 1 - rx(1 + \Delta), \\ \lambda_1 &= 1 + rx_1(1 + \Delta_1), \\ \lambda_2 &= 1 + rx_2(1 + \Delta_2), \end{aligned} \tag{E.7}$$

after which the expressions can be expanded in powers of r. However, there is a complication. If we choose $\Delta = \Delta_1 = \Delta_2 = 0$, then we obtain

$$\mathscr{R} = r^2 \mathscr{R}_0 + 2r^3 x x_1 x_2 \tag{E.8}$$

with the homogeneous polynomial

$$\mathscr{R}_0 = x^2 + x_1^2 + x_2^2 + 2xx_1 + 2xx_2 - 2x_1 x_2. \tag{E.9}$$

The last term in Eq. (E.8) makes the integration unpleasant. Therefore, we choose $\Delta, \Delta_1, \Delta_2$ such that $\mathscr{R} = r^2 \mathscr{R}_0$ holds up to the desired order in r. With

$$\begin{aligned} \Delta &= \alpha r x_1 + \beta r x_2, \\ \Delta_1 &= -\alpha r x + \gamma r x_2, \\ \Delta_2 &= -\beta r x + \gamma r x_1, \end{aligned} \tag{E.10}$$

such that $\alpha + \beta + \gamma = -\frac{1}{2}$, one then has $\mathscr{R} = r^2 \mathscr{R}_0 + O(r^4)$, which is sufficient for the calculation of the leading order and next-to-leading order terms in $\overline{|S_{ab}|^2}$. One can easily find the higher order contributions to the Δ's, so that the relation $\mathscr{R} = r^2 \mathscr{R}_0$ is fulfilled to higher orders in r.

With these substitutions, we obtain

$$\ln\left(\frac{1+\lambda}{\lambda_1 + \lambda_2}\right)^M = \frac{M'}{r}\left[\ln(1+\lambda) - \ln(\lambda_1 + \lambda_2)\right] = \frac{M'}{2}(x + x_1 + x_2) + O(r), \tag{E.11}$$

so that

$$\begin{aligned} \left(\frac{1+\lambda}{\lambda_1 + \lambda_2}\right)^M e^{-2t(\lambda_2^2 - 1)} &= e^{-\mu_1 x - \mu_1 x_1 - \mu_2 x_2} \exp(k_p), \\ \left(\frac{1+\lambda}{\lambda_1 + \lambda_2}\right)^M e^{-2t(\lambda_2^2 - \lambda^2)} &= e^{-\mu_2 x - \mu_1 x_1 - \mu_2 x_2} \exp(k_q), \end{aligned} \tag{E.12}$$

where k_p and k_q are of order $O(r^1)$, and

$$\mu_1 = M'/2, \qquad \mu_2 = M'/2 + 4t'. \tag{E.13}$$

Thus,

$$\overline{|S_{ab}|^2} = \frac{r}{2} \int_0^\infty dx \int_0^\infty dx_1 \int_0^\infty dx_2 \left[e^{-\mu_1 x - \mu_1 x_1 - \mu_2 x_2} \frac{p}{\mathscr{R}_0} + e^{-\mu_2 x - \mu_1 x_1 - \mu_2 x_2} \frac{q}{\mathscr{R}_0} \right] \tag{E.14}$$

with

$$\begin{aligned} p &= \tilde{p} J \exp(k_p), \\ q &= \tilde{q} J \exp(k_q), \\ J &= -\frac{1}{r^3} \frac{\partial(\lambda, \lambda_1, \lambda_2)}{\partial(x, x_1, x_2)} = 1 + O(r). \end{aligned} \tag{E.15}$$

The functions p and q can now be expanded in powers of r, the expansion coefficients being polynomials in x, x_1, x_2. To lowest order, one has for $\overline{|S_1|^2}$,

$$\begin{aligned} p &= \tfrac{1}{2} - \tfrac{1}{4} M'x + 2t'x + \tfrac{1}{4} M'x_1 + 2t'x_1 - \tfrac{1}{4} M'x_2 + 2t'x_2 + O(r), \\ q &= \tfrac{1}{2} - \tfrac{1}{4} M'x - 4t'x - \tfrac{1}{4} M'x_1 - 4t'x_1 + \tfrac{1}{4} M'x_2 + O(r), \end{aligned} \tag{E.16}$$

and for $\overline{|S_2|^2}$,

$$\begin{aligned} p &= 1 - \tfrac{1}{2} M'x + 2t'x + 2t'x_1 - 2t'x_2 + O(r), \\ q &= -1 + \tfrac{1}{2} M'x + 6t'x - 2t'x_1 + 2t'x_2 + O(r). \end{aligned} \tag{E.17}$$

Higher order contributions in r have been calculated using MAPLE. The expressions are lengthly and will not be reproduced here.

The integral involving the polynomial p is evaluated in the following way: The first step consists of substituting $x = s - x_1$ and integrating over x_1 from zero to s. We note that

$$\mathscr{R}_0 = (s + x_2)^2 - 4x_1 x_2 \tag{E.18}$$

is linear in x_1. This leaves us with the double integral

$$\frac{1}{2} \int_0^\infty dx_2 \int_0^\infty ds\, e^{-\mu_1 s - \mu_2 x_2} \left[\frac{p_l(s, x_2)}{2x_2} \ln \frac{s + x_2}{|s - x_2|} + p_p(s, x_2) \right], \tag{E.19}$$

where $p_l(s, x_2) = p(x = -(s - x_2)^2/(4x_2),\ x_1 = (s + x_2)^2/(4x_2),\ x_2)$ and $p_p(s, x_2)$ is the integral of $(p - p_l)/\mathscr{R}_0$ over x_1. We note that $(p - p_l)/\mathscr{R}_0$ is a polynomial in x_1. In the second step, the substitution

$$s = (1 - u)\, v/\mu_1, \qquad x_2 = uv/\mu_2 \tag{E.20}$$

yields the integral

$$\int_0^1 du \int_0^\infty dv\, e^{-v} \left[\frac{p_l}{\mu_1 u} \ln \frac{\mu_2 - (\mu_2 - \mu_1) u}{|\mu_2 - (\mu_2 + \mu_1) u|} + \frac{v}{\mu_1 \mu_2} p_p \right]. \tag{E.21}$$

Since p_l and p_p are polynomials in v, the v-integration is simple. Finally one is left with the u-integral, in which p_l and p_p are Laurent series in u with contributions from u^{-n} to u^n if p is a polynomial of order n in x, x_1, x_2. However, the total integrand is analytic at $u=0$. If the contributions including $O(r^k)$ in p are taken into account, then $n=2k+1$. A similar procedure applies to the q-integral.

Using MAPLE, the leading term for $\overline{|S_i|^2}$ and the next three sub-leading terms have been calculated, and are given by

$$\begin{aligned}\overline{|S_1|^2} = {} & \frac{4t+M}{M(8t+M)} - \frac{4t+M}{M(8t+M)^2} \\ & + \frac{2048t^4 + 1280t^3M + 512t^2M^2 + 20tM^3 + M^4}{M^2(8t+M)^5} \\ & - \frac{16384t^5 + 12288t^4M + 5376t^3M^2 + 2048t^2M^3 + 20tM^4 + M^5}{M^3(8t+M)^6},\end{aligned} \tag{E.22}$$

$$\begin{aligned}\overline{|S_2|^2} = {} & \frac{4t}{M(8t+M)} - \frac{4t}{M(8t+M)^2} \\ & + \frac{-2048t^4 - 768t^3M - 320t^2M^2 + 4tM^3}{M^2(8t+M)^5} \\ & - \frac{16384t^5 + 8192t^4M + 3328t^3M^2 - 1088t^2M^3 + 4tM^4}{M^3(8t+M)^6}.\end{aligned} \tag{E.23}$$

Although the structure of the integrals would allow the appearance of logarithms and dilogarithms, it turns out that the final integrals are rational functions of M and t. From the expressions (E.22) and (E.23), one obtains the contributions to the conductance quoted in Section 5.2.

APPENDIX F

A numerical treatment of the integrals in Eqs. (5.2), (5.4) cannot be implemented directly as their integrands exhibit an algebraic singularity on the surface

$$\mathscr{R} \equiv \lambda^2 + \lambda_1^2 + \lambda_2^2 - 2\lambda\lambda_1\lambda_2 - 1 = 0, \tag{F.1}$$

which, of course, cancels out between numerator and denominator in the end, once all relevant contributions are taken into account. This apparent singularity can be eliminated altogether via the following change of integration variables:

$$\lambda = 1 - 2q,$$
$$\lambda_1 = 1 + 2qq_1, \tag{F.2}$$
$$\lambda_2 = 1 + 2qq_2.$$

Then q, q_1, q_2 span the ranges $0 \leqslant q_1, q_2 < \infty$ and $0 \leqslant q \leqslant 1$.

Let us now consider the case $t = 0$ and write

$$\overline{|S_1|^2} = \int_0^1 dq\, I_M(q). \tag{F.3}$$

Then one can show, for example, taking $M = 1$, that $I_1(q) \sim_{q \to \infty} \text{const} \times q^{-1/2}$, so that there is an integrable singularity at $q = 0$. The divergence of $I_M(q \to 0)$ does not disappear for larger values of M. It can be removed by the transformation $q = 1/(p+1)$. The resulting integral is regular in p but displays a consequent slow falloff as $p \to \infty$, going as $p^{-3/2}$ for $M = 1$. So, to achieve an accuracy of 10^{-4}, the numerical cutoff in the p-integral would have to be taken as 10^8.

Nonetheless, we thus arrive at the expressions

$$\overline{|S_1|^2} = \int_0^\infty dp \int_0^\infty dq_1 \int_0^\infty dq_2 \left(\frac{p}{1+p+q_1+q_2}\right)^{M-1} \frac{\exp[-8tq_2(1+p+q_2)/(p+1)^2]}{(1+p+q_1+q_2)^2}$$
$$\times \frac{1}{\mathscr{R}_p}\left[1+q_1+q_2+\frac{p}{p+1}\frac{2q_1q_2}{1+p+q_1+q_2}\right]\left\{\frac{p+1}{\mathscr{R}_p}[(1+\mathscr{E})p-(1-\mathscr{E})\right.$$
$$\left.\times(q_1-q_2)(1+p+q_1+q_2)]+\frac{4t}{(p+1)^2}[4\mathscr{E}p+(1-\mathscr{E})(1+p+2q_2)^2]\right\} \tag{F.4}$$

and

$$\overline{|S_2|^2} = \int_0^\infty dp \int_0^\infty dq_1 \int_0^\infty dq_2 \left(\frac{p}{1+p+q_1+q_2}\right)^{M-1}$$
$$\times \frac{\exp[-8tq_2(1+p+q_2)/(p+1)^2]}{(1+p+q_1+q_2)^2}\frac{1}{\mathscr{R}_p}$$
$$\times\left\{-\frac{1+\mathscr{E}-\mathscr{F}}{p+1}\left[(q_1-q_2)\left[3+\frac{2p}{\mathscr{R}_p}((p+1)(1+q_1+q_2)+2q_1q_2)\right]\right.\right.$$

$$+\frac{4t}{(p+1)^2}[2p((p+1)(1-q_1+q_2)+2q_2)-(1+p+q_1+q_2)(1+p+q_2)^2]\Bigg]$$

$$+\frac{(1-\mathscr{E})p}{p+1}\Bigg[\Bigg[1+\frac{2}{\mathscr{R}_p}((p+1)(1+q_1+q_2)+2q_1q_2)\Bigg](1+p+q_1+q_2)$$

$$+\frac{4t}{(p+1)^2}[2(p+1)(1-q_1+q_2)+4q_1+(q_1-q_2)(1+p+2q_2)^2]\Bigg]\Bigg\}. \quad (F.5)$$

We have set $\mathscr{R}_p \equiv (p+1)\mathscr{R}$ so that

$$\mathscr{R}_p=(p+1)(1+q_1+q_2)^2-4pq_1q_2, \quad (F.6)$$

and here $\mathscr{E}=\exp[-8/(p+1)^2]$. In this form, the integration is completely free of singularities; but, for small t, the integrand decays slowly (with a power law) in all directions. Consequently, all three upper integration limits must be taken very large, and this makes accurate computation slow and difficult.

To circumvent this, we introduce two compact coordinates by transforming to two angle variables α, w and one radial parameter r according to the definitions

$$\begin{aligned} q_1 &= \tfrac{1}{2}r(1-\alpha)(1+w), \\ q_2 &= \tfrac{1}{2}r(1-\alpha)(1-w), \\ p &= r\alpha. \end{aligned} \quad (F.7)$$

The ranges of the new variables are given by $0\leqslant r\leqslant\infty$, $0\leqslant\alpha\leqslant 1$, $-1\leqslant w\leqslant 1$. However, because of the very asymmetric form that the exponential now acquires, there exist both rapidly and very slowly decaying directions for the integrand. Indeed, the upper limit of the r-integral must still be taken extremely large in order to achieve sufficient accuracy for the slowly decaying parts. However, this means that the integration region also extends far along the rapidly decaying directions where the integrand is virtually zero; and, in fact, the integrand turns out to have support only in a tiny fraction of the entire integration volume. The integration routine is then unable to properly determine when the desired accuracy has been achieved. This is because, at some state in the iteration, further subdivision of the grid is likely to create new points essentially only in the pervading insignificant region and, hence, not alter the previous value of the integral to within the requisite accuracy.

The easisest way to remedy this problem is by compactifying the r-integration through the variable change

$$y=\frac{r}{1+r}, \quad (F.8)$$

so that $0\leqslant y\leqslant 1$. The transformations of Eqs. (F.7), (F.8) result in a simple bounded integration over a hyper-rectangular region and an integrand that has a significant value over an appreciable fraction of this region. The change of integration measure and region that corresponds to the variable transformation $(p, q_1, q_2) \to (y, \alpha, w)$ can be summarized by the equation

$$\int_0^\infty dp \int_0^\infty dq_1 \int_0^\infty dq_2 \left(\frac{p}{1+p+q_1+q_2}\right)^{M-1} \frac{1}{(1+p+q_1+q_2)^2}[\cdots]$$
$$=\frac{1}{2}\int_0^1 dy \frac{y^{M+1}}{(1-y)^2}\int_0^1 d\alpha\,(1-\alpha)\,\alpha^{M-1}\int_{-1}^{+1} dw\,[\cdots]. \tag{F.9}$$

Then we can write $\overline{|S_j|^2}$ $(j=1, 2)$,

$$\overline{|S_j|^2}=\frac{1}{2}\int_0^1 dy\, y^{M+1}\int_0^1 d\alpha(1-\alpha)\,\alpha^{M-1}\int_{-1}^{+1} dw\,\frac{1-y}{\mathscr{R}_y}$$
$$\times \exp\left\{-2t\frac{y(1-\alpha)(1-w)}{1-y(1-\alpha)}\left[2+\frac{y(1-\alpha)(1-w)}{1-y(1-\alpha)}\right]\right\} F_j(y, \alpha, w), \tag{F.10}$$

where

$$F_1(y, \alpha, w)=\frac{1}{1-y}\left[1-y\alpha-\frac{1}{2}\frac{y^3\alpha(1-\alpha)^2(1-w^2)}{1-y(1-\alpha)}\right]$$
$$\times\left\{\frac{y(1-y(1-\alpha))}{\mathscr{R}_y}\left[(1+\mathscr{E})(1-y)\,\alpha-(1-\mathscr{E})(1-\alpha)\,w\right]\right.$$
$$\left.+4t\left[4\mathscr{E}\frac{y(1-y)\,\alpha}{[1-y(1-\alpha)]^2}+(1-\mathscr{E})\left(1+\frac{y(1-\alpha)(1-w)}{1-y(1-\alpha)}\right)^2\right]\right\} \tag{F.11}$$

and

$$F_2(y, \alpha, w)=-(1+\mathscr{E}-\mathscr{F})\left\{\frac{y(1-\alpha)\,w}{1-y(1-\alpha)}\left[3+\frac{2y\alpha}{\mathscr{R}_y}\left((1-y)+y^2\alpha(1-\alpha)\right.\right.\right.$$
$$\left.\left.+\frac{1}{2}y^2(1-\alpha)^2(1-w^2)\right)\right]$$
$$\left.+\frac{4t}{[1-y(1-\alpha)]^3}\left[2y\alpha(1-y-y^2\alpha(1-\alpha)\,w)-(1-y(1-\alpha)\,w)^2\right]\right\}$$
$$+(1-\mathscr{E})\frac{y\alpha}{1-y(1-\alpha)}\left\{\frac{1}{1-y}+\frac{1}{\mathscr{R}_y}\left[2(1-y)+2y^2\alpha(1-\alpha)\right.\right.$$
$$\left.+y^2(1-\alpha)^2(1-w)^2\right]+\frac{4t}{1-y}\frac{1}{[1-y(1-\alpha)]^2}\left[2(1-y)(1-y\right.$$
$$\left.\left.-y^2\alpha(1-\alpha)\,w)+y(1-\alpha)\,w(1-y(1-\alpha)\,w)^2\right]\right\}, \tag{F.12}$$

with

$$\mathcal{R}_y = y^2(1-y)(1-\alpha)^2 + (1-y)[1-y(1-\alpha)][1-y+2y(1-\alpha)] + y^3\alpha(1-\alpha)\,w^2,$$

$$\mathcal{E} = \exp\left\{-8t\frac{y(1-y)\,\alpha}{[1-y(1-\alpha)]^2}\right\},$$

$$\mathcal{F} = \frac{1-e^{-8tx}}{4tx}, \qquad x = \frac{y(1-y)\,\alpha}{[1-y(1-\alpha)]^2}. \tag{F.13}$$

In the numerical analysis, we represent $\mathcal{F}$ as the integral

$$\mathcal{F}(4tx) = 2\int_0^1 du e^{-8txu} \tag{F.14}$$

when the argument is small. It is also useful to note the relation

$$\frac{1}{\mathcal{R}_p} = \frac{(1-y)^3}{\mathcal{R}_y}. \tag{F.15}$$

In the parameterization above, a mild algebraic singularity appears along the line $y=1$, $w=0$. This can be easily dealt with by setting the upper limit of the y-integration to $1-\delta$ with δ an arbitrarily small positive number.

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References

1. S. Iida, H. A. Weidenmüller, and J. A. Zuk, *Ann. Phys.* (*N.Y.*) **200** (1990), 219.
2. A. D. Mirlin, A. Müller-Groeling, and M. R. Zirnbauer, *Ann. Phys.* (*N.Y.*) **236** (1994), 325.
3. J. A. Zuk, *Phys. Rev. B* **45** (1992), 8952.
4. A. Müller-Groeling, *Phys. Rev. B* **47** (1993), 6480.
5. A. Altland, S. Iida, A. Müller-Groeling, and H. A. Weidenmüller, *Ann. Phys.* (*N.Y.*) **219** (1992), 148.
6. C. M. Marcus, A. J. Westervelt, P. F. Hopkins, and A. C. Gossard, *Phys. Rev. Lett.* **69** (1992), 506.
7. M. J. Berry, J. H. Baskey, R. M. Westervelt, and A. C. Gossard, *Surf. Sci.* **305** (1994), 495.
8. M. W. Keller, O. Millo, A. Mittal, and D. E. Prober, *Surf. Sci.* **305** (1994), 501.
9. H. U. Baranger and P. A. Mello, *Phys. Rev. Lett.* **73** (1994), 142.
10. C. H. Lewenkopf and H. A. Weidenmüller, *Ann. Phys.* (*N.Y.*) **212** (1991), 53.

11. A. Pandey and M. L. Mehta, *Comm. Math. Phys.* **87** (1983), 449.
12. H. U. Baranger, R. A. Jalabert, and A. D. Stone, *Phys. Rev. Lett.* **70** (1993), 3876.
13. C. Mahaux and H. A. Weidenmüller, "Shell-Model Approach to Nuclear Reactions," North-Holland, Amsterdam, 1969.
14. J. J. M. Verbaarschot, H. A. Weidenmüller, and M. R. Zirnbauer, *Phys. Rep.* **129** (1985), 367.
15. K. B. Efetov, *Adv. Phys.* **32** (1983), 53.
16. A. Pandey and M. L. Mehta, *J Phys. A* **16** (1983), 2655, L601.
17. A. M. Ozório de Almeida, *in* "Quantum Chaos" (H. A. Cerdeira, R. Ramaswamy, M. C. Gutzwiller, and G. Casati, Eds.), World Scientific, Singapore, 1991.
18. M. V. Berry, *in* "Semiclassical Mechanics of Regular and Irregular Motion in Chaotic Behaviour of Deterministic Systems" (G. Loos, R. Helleman, and R. Stora, Eds.), North-Holland, Amsterdam, 1983.
19. E. J. Heller, P. W. O'Connor, and J. Gehlen, *Phys. Scr.* **40** (1989), 354.
20. M. L. Mehta, "Random Matrices," 2nd ed., Academic Press, New York, 1991.
21. M. V. Berry and M. Robnik, *J. Phys. A* **19** (1986), 649.
22. M. R. Zirnbauer, private communication.
23. A. Altland, S. Iida, and K. B. Efetov, *J. Phys. A* **26** (1993), 3545.
24. J. J. M. Verbaarschoot and M. R. Zirnbauer, *J. Phys. A* **17** (1985), 1093.
25. A. Pruisken and L. Schäfer, *Nucl. Phys. B* **200** (1982), 20.
26. B. L. Altshuler, A. G. Aronov, and B. Z. Spivak, *JETP Lett.* **33** (1981), 94.
27. R. Jensen, *Chaos* **1** (1991), 101.
28. E. Doron, U. Smilansky, and A. Frenkel, *Physica D* **50** (1991), 367.
29. J. F. Reading, *Phys. Rev. A* **8** (1973), 3262.
30. S. Chakravarty and A. Schmid, *Phys Rep.* **140** (1986), 193.
31. H. Graef *et al.*, *Phys. Rev. Lett.* **69** (1992), 1296.