Anyons

The argument that we have used to describe exchange symmetry is, in fact, only strictly valid in three dimensions. In two dimensions, there are further possibilities other than fermions and bosons. For the interested reader, we give a more detailed description in this box.

We begin by noticing that eqn 29.3 allows the solution $\psi(\boldsymbol{r}_2, \boldsymbol{r}_1) = e^{i\theta}\psi(\boldsymbol{r}_1, \boldsymbol{r}_2)$, where θ is a phase factor. Thus exchanging identical particles means that the wave function acquires a phase θ. Defining $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$, the action of exchanging the position coordinates of two particles involves letting this vector execute some path from $\boldsymbol{r}$ to $-\boldsymbol{r}$, but avoiding the origin so that the two particles do not ever occupy the same position.

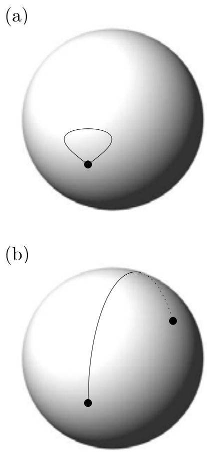

Fig. 29.1 Paths in $\boldsymbol{r}$-space, for the three-dimensional case, corresponding to (a) no exchange of particles and (b) exchange of particles.

We therefore can imagine the exchange of particles as a path in $\boldsymbol{r}$-space. Without loss of generality, we can keep $|\boldsymbol{r}|$ fixed, so that in the process of exchanging the two particles, they move relative to each other at a fixed separation. Thus, for the case of three dimensions, the path is on the surface of a sphere in $\boldsymbol{r}$-space. Since the two particles are identical, opposite points on the surface of the sphere are equivalent and must be identified (giving $\boldsymbol{r}$-space the topology of, what is known as, real two-dimensional projective space). It turns out that all paths on this surface fall into two classes: ones which are contractible to a point [and thus correspond to no exchange of particles, yielding $\theta = 0$ to ensure the wavefunction is single-valued; see Fig. 29.1(a)] and those which are not [and thus correspond to exchange of particles; see Fig. 29.1(b)]. For this latter case we have to assign $\theta = \pi$, so that two exchanges correspond to no exchange, i.e. $e^{i\theta}e^{i\theta} = 1$, so that $\theta = \pi$. This argument thus justifies that the phase factor $e^{i\theta} = \pm 1$, giving rise to bosons ($e^{i\theta} = +1$) and fermions ($e^{i\theta} = -1$).

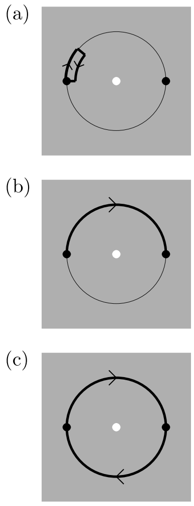

Fig. 29.2 Paths in $\boldsymbol{r}$-space, for the two-dimensional case, corresponding to (a) no exchange, (b) a single exchange and (c) two exchanges of particles.

However, the argument fails in two dimensions. In the two-dimensional case, the path is on a circle in $\boldsymbol{r}$-space in which opposite points on the circle are equivalent and are identified. In this case, the paths in $\boldsymbol{r}$-space can wind round the origin an integer number of times. This means that two successive exchanges of the particles [as shown in Fig. 29.2(c)] are not topologically equivalent to zero exchanges [if performed by winding round the origin in the same direction, as shown in Fig. 29.2(a)] and thus the phase θ can take any value. (In this case, $\boldsymbol{r}$-space has the topology of real one-dimensional projective space, which is the same as that of a circle.) The resulting particles have more complicated statistical properties than either bosons or fermions and are called **anyons** (because θ can take 'any' value). Since θ/π is no longer forced to be ± 1, and can take any *fractional* value in between, anyons can have **fractional statistics**. The crucial distinction between $\boldsymbol{r}$-space in two and three dimensions is that the removal of the origin in two-dimensional space makes the space multiply connected (allowing paths which wind around the origin), whereas three-dimensional space remains singly connected (and a path which tries to wind round the origin can be deformed into one which does not).

We live in a three-dimensional world, so is any of this relevant? In fact, anyons turn out to be important in the **fractional quantum Hall effect**, which occurs in certain two-dimensional electron systems under high magnetic field. For more details concerning anyons, see the further reading.